CSC 8301- Design and Analysis of Algorithms

Lecture 6

Divide and Conquer
Algorithm Design Technique

Divide-and-Conquer

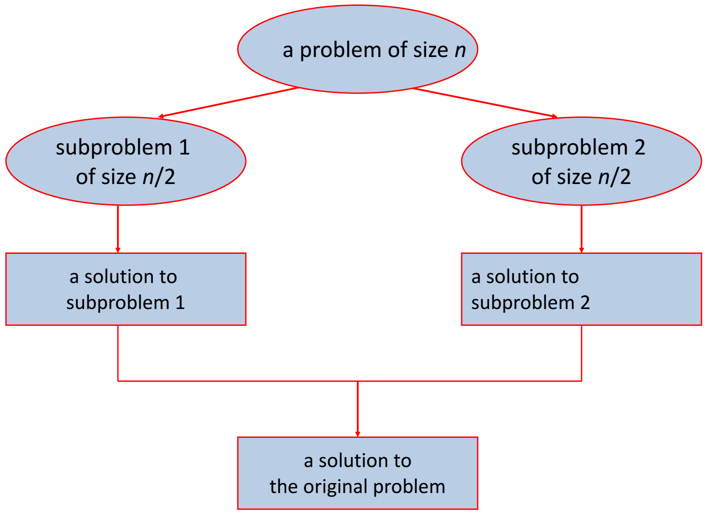
The most-well known algorithm design strategy:

1. Divide a problem instance into two or more smaller instances (ideally of about the same size)

2. Solve the smaller instances (usually recursively)

3. Obtain a solution to the original instance by combining these solutions to the smaller instances

Divide-and-Conquer Technique (cont.)



Divide-and-Conquer Examples

General Divide-and-Conquer Recurrence

$T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^d)$, $d \geq 0$

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Master Theorem: If $a < b^d$, $T(n) \in \Theta(n^d)$
If $a = b^d$, $T(n) \in \Theta(n^d \log n)$
If $a > b^d$, $T(n) \in \Theta(n^{\log_b a})$

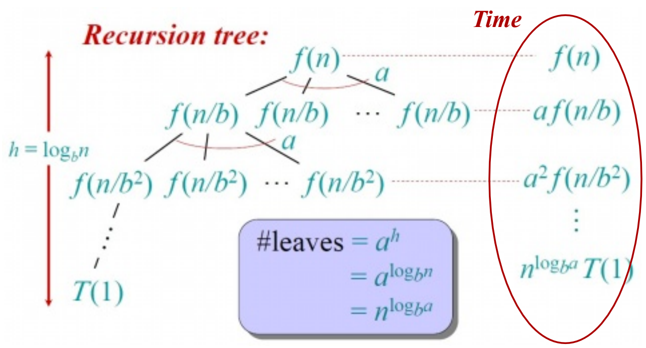
Note: The same results hold with O instead of Θ

Examples: $T(n) = 4T(n/2) + n \Rightarrow T(n) \in ?$
$T(n) = 4T(n/2) + n^2 \Rightarrow T(n) \in ?$
$T(n) = 4T(n/2) + n^3 \Rightarrow T(n) \in ?$

Master Theorem – Recursion Tree

$T(n) = aT(n/b) + f(n)$

Visualize this as a recursion tree (branch factor a):



Total time depends on how fast *f(n)* grows compared with the number of leaves (d compared with $\log_b a$)

Divide-and-Conquer Examples

- Sorting: mergesort and quicksort
- Binary tree traversals
- Binary search (?)
- Multiplication of large integers
- Closest-pair algorithm

Mergesort

- Split array A[0..n-1] in two about equal halves and make copies of each half in arrays B and C
- Sort arrays B and C recursively
- Merge sorted arrays B and C into array A as follows:
  - Repeat until no elements remain in one of the arrays:
    - compare the first elements in the remaining unprocessed portions of the arrays
    - copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array
  - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into A

Mergesort Example

8 3 2 9 7 1 5 4

Merging of Two Sorted Arrays

ALGORITHM $Merge(B[0..p-1], C[0..q-1], A[0..p+q-1])$
//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0$; $j \leftarrow 0$; $k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
  if $B[i] \leq C[j]$
    $A[k] \leftarrow B[i]$; $i \leftarrow i+1$
  else $A[k] \leftarrow C[j]$; $j \leftarrow j+1$
  $k \leftarrow k+1$
if $i = p$
  copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

Analysis of Mergesort

- Recurrence for number of comparisons in the worst case:

  $C(n) = 2\,C(n/2) + C_{merge}(n) = 2\,C(n/2) + n-1$

  $C(1) = 0$

Analysis of Mergesort

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  $C(n) \;\; = 2\,C(n/2) + C_{merge}(n) = 2\,C(n/2) + n\text{-}1$

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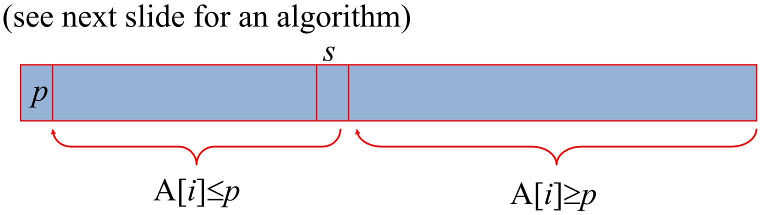
- All cases have same efficiency: $\Theta(n \log n)$

- Space requirement: $\Theta(n)$ (<u>not</u> in-place)

- Can be implemented without recursion (bottom-up)

Quicksort

- Select a *pivot* (partitioning element) – here, the first element
- Rearrange the list so that all the elements in the first *s* positions are smaller than or equal to the pivot and all the elements in the remaining *n-s* positions are larger than or equal to the pivot (see next slide for an algorithm)



- Exchange the pivot with the last element in the first (i.e., ≤) subarray — *the pivot is now in its final position*
- Sort the two subarrays recursively

Two-Way (Hoar's) Partitioning Algorithm

```
Algorithm Partition(A[l..r])
//Partitions a subarray by using its first element as a pivot
//Input: A subarray A[l..r] of A[0..n − 1], defined by its left and right
//        indices l and r (l < r)
//Output: A partition of A[l..r], with the split position returned as
//         this function's value
p ← A[l]
i ← l;  j ← r + 1
repeat
    repeat i ← i + 1 until A[i] ≥ p
    repeat j ← j − 1 until A[j] ≤ p
    swap(A[i], A[j])
until i ≥ j
swap(A[i], A[j])   //undo last swap when i ≥ j
swap(A[l], A[j])
return j
```

Quicksort Example

8 3 2 9 7 1 5 4

Analysis of Quicksort

- Best-case time efficiency: split in the middle — $\Theta(n \log n)$

- Worst-case time efficiency: sorted array! — $\Theta(n^2)$

- Average case time efficiency: random arrays — $\Theta(n \log n)$

Analysis of Quicksort

- Not stable
- Space efficiency: $\Theta(\log n)$ on average (recursive calls)

- Improvements:
  - better pivot selection: median-of-three partitioning
  - stop recursive calls when unsorted subarrays become small (say, <10 elements) and finish sorting with insertion sort

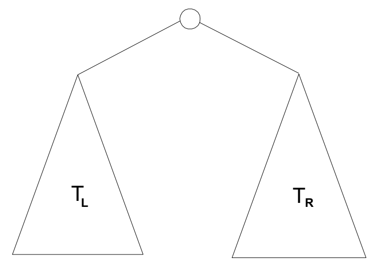
  These yields about 20% improvement

- Considered the method of choice for sorting random files of nontrivial sizes

Binary Tree Algorithms

Binary tree is a divide-and-conquer ready structure!

Example 1: Classic traversals
- *Preorder*: root, left, right
- *Inorder*: left, root, right
- Postorder: left, right, root



Algorithm *Inorder*(T)

if $T \neq \varnothing$

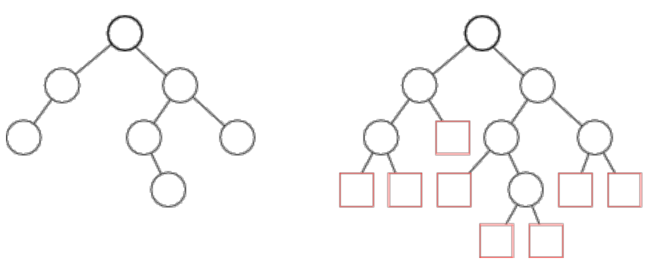
  Inorder(T_{left})

  print(root of T)

  Inorder(T_{right})

Extended Binary Tree

- Replace every null subtree of the original tree with special extra nodes (called *extended* or *external*)

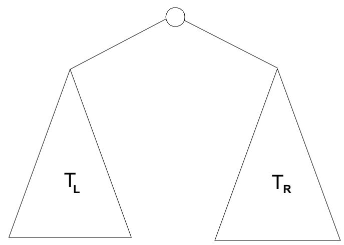


- How many extended nodes in a tree with n (original) nodes?

Binary Tree Algorithms (cont.)

Example 2: Computing the height of a binary tree

- The height is the length of the longest path (counting edges) on the way from the root to a leaf
- The height of a single node is 0



$h(T) = \max\{h(T_L), h(T_R)\} + 1$ if $T \neq \varnothing$ and $h(\varnothing) = -1$

Efficiency: $\Theta(n)$

Multiplication of Large Integers

Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

The grade-school algorithm:

$$
\begin{array}{r}
a_1\ a_2 \ldots\ a_n \\
\underline{b_1\ b_2 \ldots\ b_n} \\
(d_{10})\ d_{11}d_{12} \ldots\ d_{1n} \\
(d_{20})\ d_{21}d_{22} \ldots\ d_{2n} \quad \\
\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\
(d_{n0})\ d_{n1}d_{n2} \ldots\ d_{nn} \qquad\quad
\end{array}
$$

Efficiency: n^2 one-digit multiplications

First Divide-and-Conquer Algorithm

A small example: $A * B$ where $A = 2135$ and $B = 4014$

$A = (21 \cdot 10^2 + 35)$, $B = (40 \cdot 10^2 + 14)$

So, $A * B = (21 \cdot 10^2 + 35) * (40 \cdot 10^2 + 14)$

$= 21 * 40 \cdot 10^4 + (21 * 14 + 35 * 40) \cdot 10^2 + 35 * 14$

In general, if $A = A_1A_2$ and $B = B_1B_2$ (where A and B are n-digit, A_1, A_2, B_1, B_2 are $n/2$-digit numbers),

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

Recurrence for the number of one-digit multiplications $M(n)$:

$$M(n) = 4M(n/2), \quad M(1) = 1$$

Solution: $M(n) = n^2$

Second Divide-and-Conquer Algorithm

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

The idea is to decrease the number of multiplications from 4 to 3:

$$(A_1 + A_2) * (B_1 + B_2) = A_1 * B_1 + (\mathbf{A_1 * B_2 + A_2 * B_1}) + A_2 * B_2,$$

I.e., $(\mathbf{A_1 * B_2 + A_2 * B_1}) = (A_1 + A_2) * (B_1 + B_2) - A_1 * B_1 - A_2 * B_2,$
which requires only 3 multiplications at the expense of (4-1) extra add/sub.

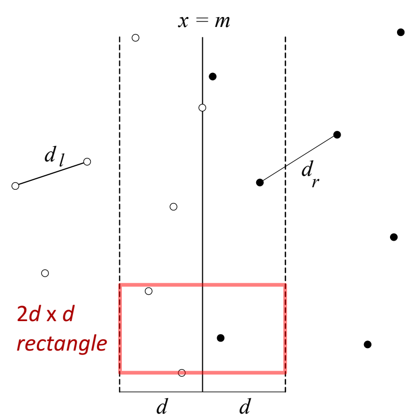
Recurrence for the number of multiplications $M(n)$:

$$M(n) = 3M(n/2), \quad M(1) = 1$$

Solution: $M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.585}$

Closest-Pair Problem by Divide-and-Conquer

Step 1 Divide the points given into two subsets P_l and P_r by a vertical line $x = m$ so that half the points lie to the left or on the line and half the points lie to the right or on the line.



$\boldsymbol{d = \min\{d_l, d_r\}}$

Closest Pair by Divide-and-Conquer (cont.)

Step 2 Find recursively the closest pairs for the left and right subsets.

Step 3 Set $d = \min\{d_l, d_r\}$

We can limit our attention to the points in the symmetric vertical strip S of width $2d$ as possible closest pair. (The points are stored and processed in increasing order of their y coordinates.)

Step 4 Scan the points in the vertical strip S from the lowest up. For every point $p(x,y)$ in the strip, inspect points in in the strip that may be closer to p than d. There can be no more than 5 such points following p on the strip list!

Efficiency of the Closest-Pair Algorithm

Recurrence for the Running time of the algorithm

$$T(n) = 2T(n/2) + M(n), \text{ where } M(n) \in O(n)$$

By the Master Theorem (with $a = 2$, $b = 2$, $d = 1$)

$$T(n) \in O(n \log n)$$

Homework

Reading: Chapter 5, Appendix B (pp. 487-491)

Exercises:

- 5.1: 1, 2, 3, 6, 8
- 5.2: 1, 7, 8, 9
- 5.3: 1, 2, 5, 8
- 5.4: 2, 3
- 5.5: 2

Next: Transform and Conquer (Ch. 6)