10. Boolean Formulas and Normal Forms

**Boolean variables** $x, y, \ldots$ take one of the two values 0 (*false*) or 1 (*true*).

**Boolean operations:** $\neg$ (NOT), $\land$ (AND), $\lor$ (OR). We write $\bar{A}$ for $\neg A$.

**Boolean formulas** are constructed from variables and operations in the standard way. Once a truth assignment for variables is given, the value of a compound formula is calculated as follows:

<table>
<thead>
<tr>
<th>$\bar{0}$</th>
<th>$0 \land 0$</th>
<th>$0 \lor 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{1}$</td>
<td>$0 \land 1$</td>
<td>$0 \lor 1$</td>
</tr>
<tr>
<td></td>
<td>$1 \land 0$</td>
<td>$1 \lor 0$</td>
</tr>
<tr>
<td></td>
<td>$1 \land 1$</td>
<td>$1 \lor 1$</td>
</tr>
</tbody>
</table>

- We say that a Boolean formula is **satisfiable** iff there is an assignment of 0s and 1s to its variables that makes the formula evaluate to 1.

- $SAT = \{\phi | \phi$ is a satisfiable Boolean formula}$

1) Evaluate: $(y \land (\bar{x} \lor \bar{y})) \lor (x \lor \bar{y})$ for $x = 0, y = 1$

2) Are the following formulas satisfiable?

- $\bar{x} \land (x \lor y)$

- $\bar{x} \land (x \land y)$
A literal is a Boolean variable $x$ or a negated Boolean variable $\bar{x}$. A clause is several literals connected with $\lor$s, as in $(x \lor y \lor z \lor t)$.

A Boolean formula is in conjunctive normal form, called a cnf-formula, if it comprises several clauses connected with $\land$s, as in $(x \lor \bar{y} \lor \bar{z} \lor t) \land (\bar{x} \lor z) \land (x \lor y \lor \bar{t})$

A cnf-formula is a 3cnf-formula if all the clauses have 3 literals, as in $(x \lor y \lor z) \land (\bar{x} \lor z \lor t) \land (x \lor y \lor \bar{t}) \land (z \lor y \lor \bar{t})$

• 3SAT = \{ $\phi$ | $\phi$ is a satisfiable 3cnf-formula}$

1) Is $(x \lor \bar{y} \lor \bar{z} \lor t) \land (\bar{x} \lor z) \land (x \lor y \lor \bar{t})$ satisfiable?

2) Convert to an equivalent cnf-formula: $(y \land (\bar{x} \lor \bar{y})) \lor (x \lor \bar{y})$

3) Define a dual notion of cnf: disjunctive normal form (dnf) and the corresponding 3-dnf

4) Give an example of a dnf formula and a 3-dnf formula

5) What do you think:
• SAT $\in$ P
• SAT $\in$ NP
• 3SAT $\in$ P
• 3SAT $\in$ NP
• What if 3SAT were defined in terms of dnf, instead of cnf?