1.5 Union Find

- dynamic connectivity
- quick find
- quick union
- improvements
- applications

Subtext of today's lecture (and this course)

- Steps to developing a usable algorithm.
  - Model the problem.
  - Find an algorithm to solve it.
  - Fast enough? Fits in memory?
  - If not, figure out why.
  - Find a way to address the problem.
  - Iterate until satisfied.

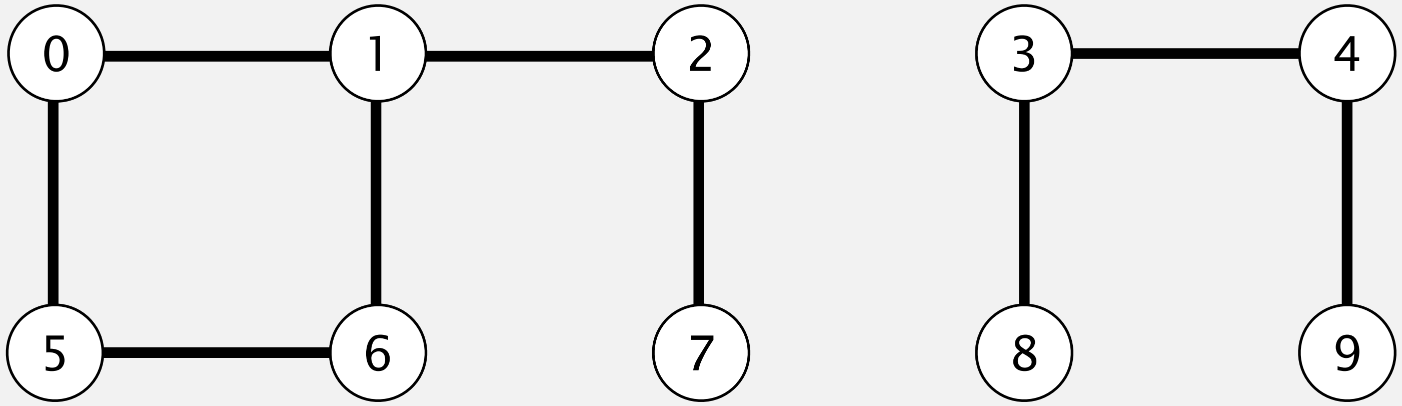
- The scientific method.

- Mathematical analysis.

Dynamic connectivity

- Given a set of N objects.
  - Union command: connect two objects.
  - Find/connected query: is there a path connecting the two objects?

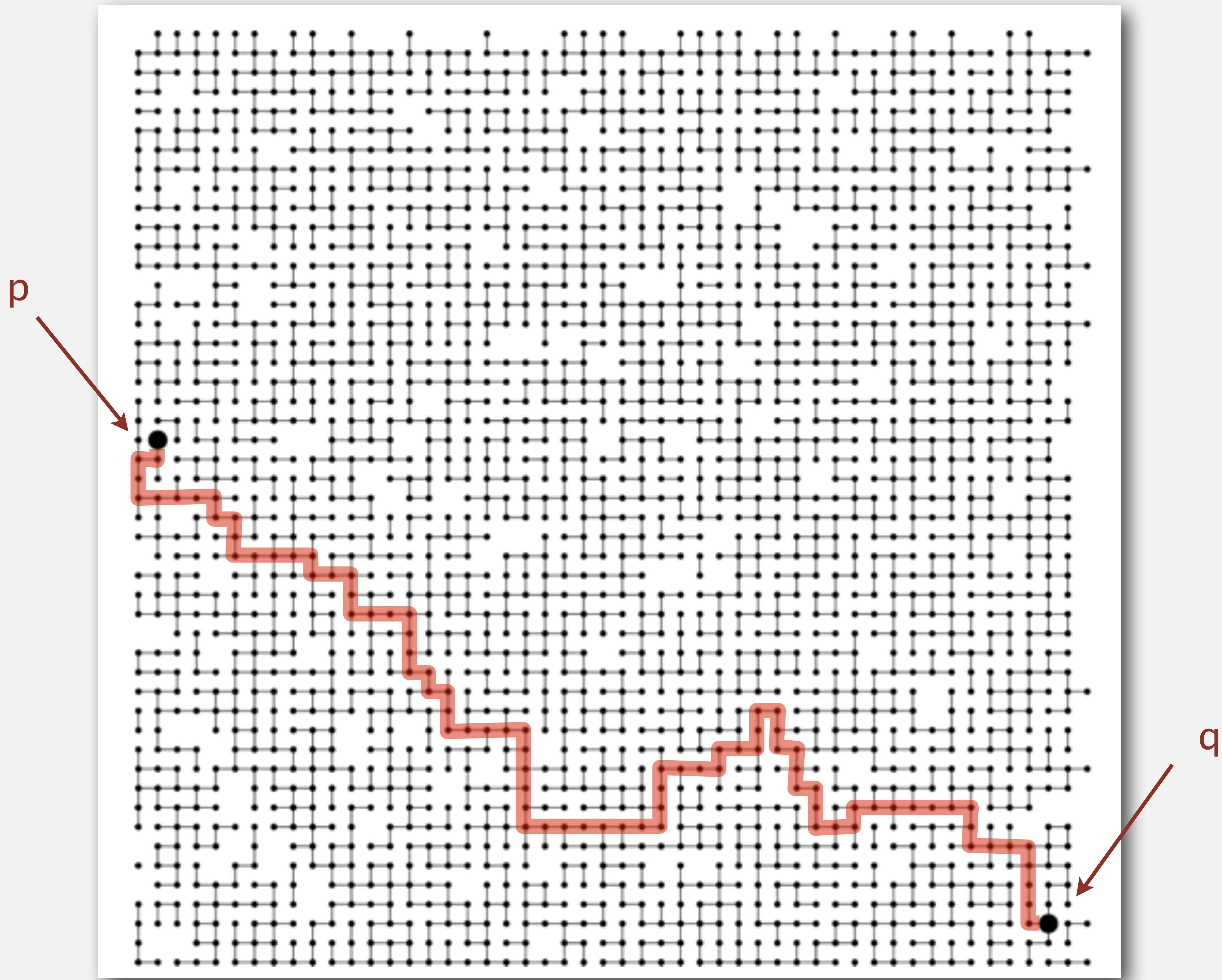
```
union(4, 3)
union(3, 8)
union(6, 5)
union(9, 4)
union(2, 1)
connected(0, 7)    ×
connected(8, 9)    ✔
union(5, 0)
union(7, 2)
union(6, 1)
connected(0, 7)    ✔
union(1, 0)
```



Connectivity example

Q. Is there a path connecting p and q ?

more difficult problem: find the path



A. Yes.

Modeling the objects

- Dynamic connectivity applications involve manipulating objects of all types.
  - Pixels in a digital photo.
  - Computers in a network.
  - Friends in a social network.
  - Transistors in a computer chip.
  - Elements in a mathematical set.
  - Variable names in Fortran program.
  - Metallic sites in a composite system.

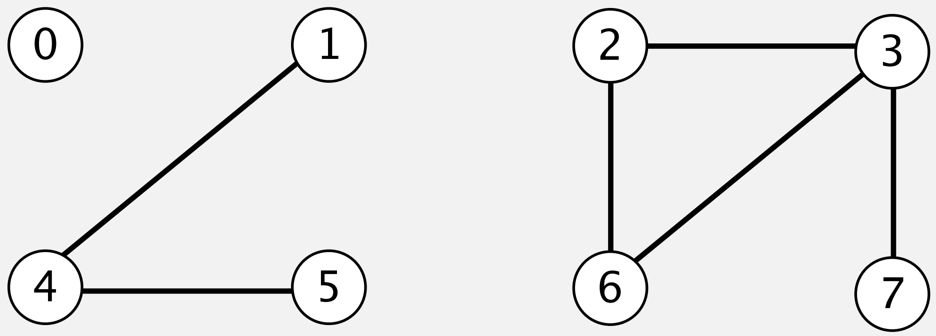
- When programming, convenient to name sites 0 to N-1.
  - Use integers as array index.
  - Suppress details not relevant to union-find.

can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

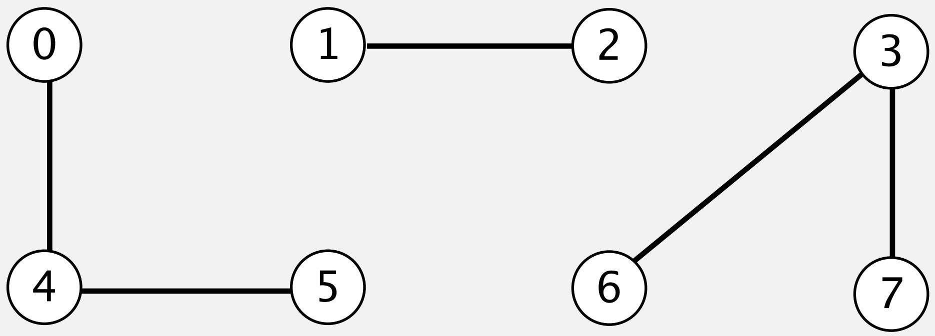
- We assume "is connected to" is an equivalence relation:
  - Reflexive: p is connected to p.
  - Symmetric: if p is connected to q, then q is connected to p.
  - Transitive: if p is connected to q and q is connected to r, then p is connected to r.

- Connected components. Maximal set of objects that are mutually connected.



{ 0 } { 1 4 5 } { 2 3 6 7 }

3 connected components

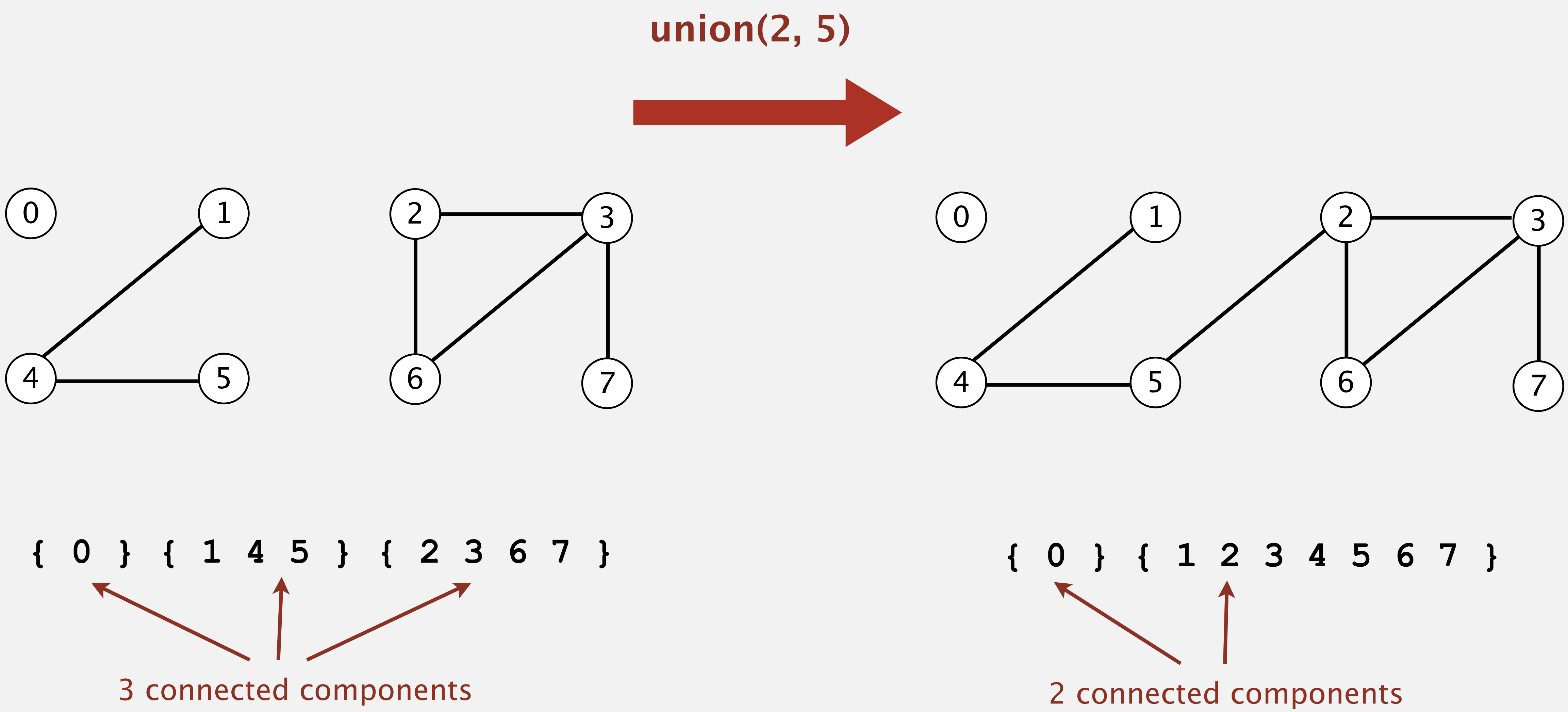


connected components ??

Implementing the operations

- **Find query.** Check if two objects are in the same component.

- **Union command.** Replace components containing two objects with their union.



Union-find data type (API)

- **Goal.** Design efficient data structure for union-find.
  - Number of objects N can be huge.
  - Number of operations M can be huge.
  - Find queries and union commands may be intermixed.

`public class UF`	
`UF(int N)`	*initialize union-find data structure with N objects (0 to $N-1$)*
`void union(int p, int q)`	*add connection between p and q*
`boolean connected(int p, int q)`	*are p and q in the same component?*
`int find(int p)`	*component identifier for p (0 to $N-1$)*
`int count()`	*number of components*

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
- read in pair of integers from standard input
- if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (!uf.connected(p, q))
      {
         uf.union(p, q);
         StdOut.println(p + " " + q);
      }
   }
}
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

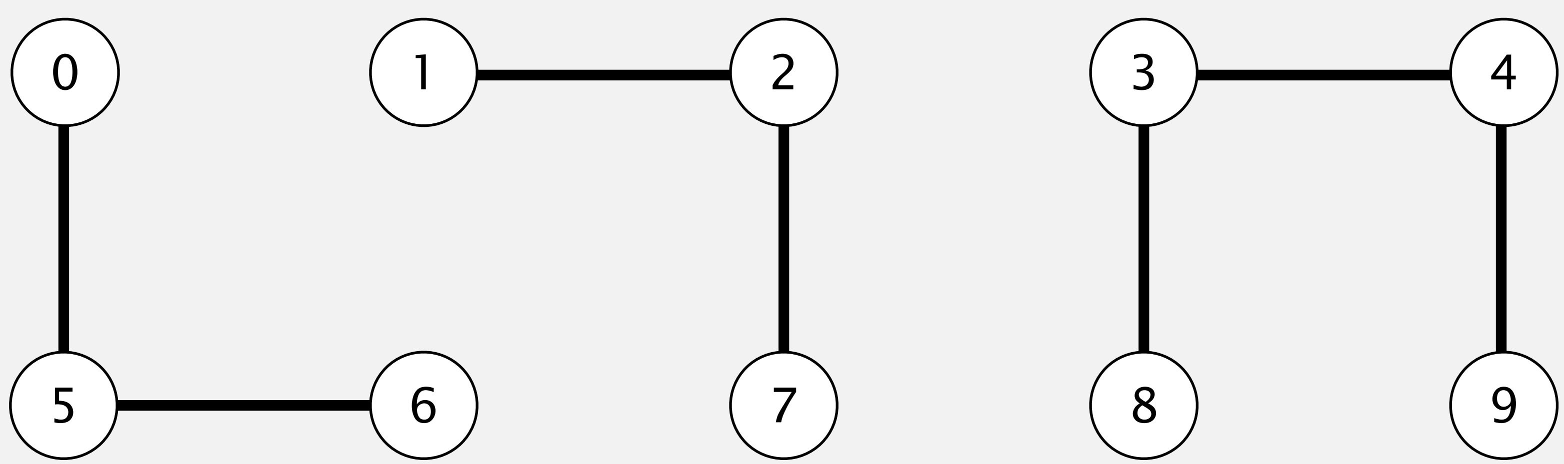
- dynamic connectivity
- **quick find**
- quick union
- improvements
- applications

Quick-find [eager approach]

- Data structure.
  - Integer array `id[]` of size `N`.
  - Interpretation: `p` and `q` are connected if and only if they have the same id.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Quick-find [eager approach]

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	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

- Find. Check if `p` and `q` have the same id.

`id[6] = 0; id[1] = 1`

6 and 1 are not connected

- Union. To merge components containing `p` and `q`, change all entries whose id equals `id[p]` to `id[q]`.

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

problem: many values can change

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean connected(int p, int q)
   {  return id[p] == id[q];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

check whether **`p`** and **`q`**
are in the same component
(2 array accesses)

change all entries with **`id[p]`** to **`id[q]`**
(at most 2N + 2 array accesses)

Quick-find is too slow

- **Cost model.** Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	N	N	1

order of growth of number of array accesses

- **Quick-find defect.** Union too expensive.

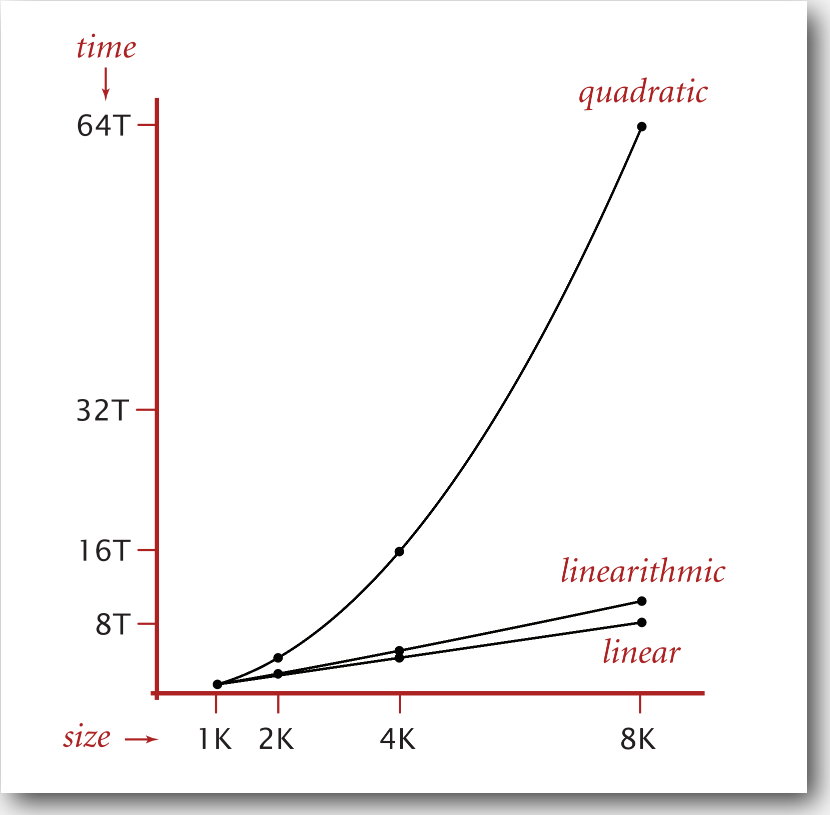
quadratic

- **Ex.** Takes N^2 array accesses to process sequence of N union commands on N objects.

Quadratic algorithms do not scale

- Rough standard (for now).
  - 10^9 operations per second.
  - 10^9 words of main memory.
  - Touch all words in approximately 1 second.

a truism (roughly) since 1950!



- Ex. Huge problem for quick-find.
  - 10^9 union commands on 10^9 objects.
  - Quick-find takes more than 10^{18} operations.
  - 30+ years of computer time!

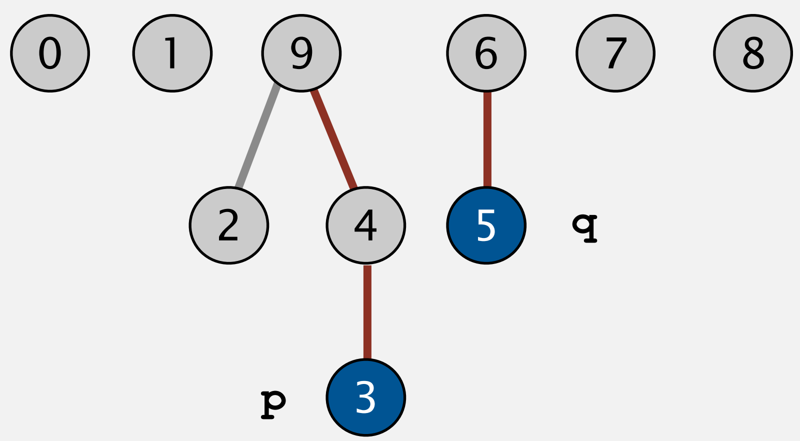
- Quadratic algorithms don't scale with technology.
  - New computer may be 10x as fast.
  - But, has 10x as much memory $\Rightarrow$ want to solve a problem that is 10x as big.
  - With quadratic algorithm, takes 10x as long!

Quick-union [lazy approach]

- Data structure.
  - Integer array `id[]` of size `N`.
  - Interpretation: `id[i]` is parent of `i`.
  - Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



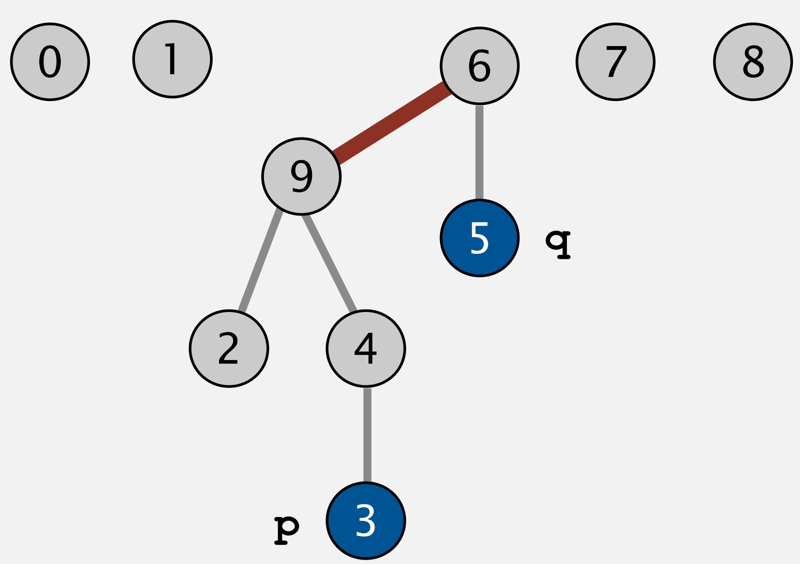
3's root is 9; 5's root is 6
3 and 5 are not connected

- Find. Check if `p` and `q` have the same root.

- Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



Quick-union demo

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   private int root(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }

   public boolean connected(int p, int q)
   {
      return root(p) == root(q);
   }

   public void union(int p, int q)
   {
      int i = root(p);
      int j = root(q);
      id[i] = j;
   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

check if p and q have same root
(depth of p and q array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

- **Cost model.** Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	N	N	1
quick-union	N	N †	N

← worst case

† includes cost of finding roots

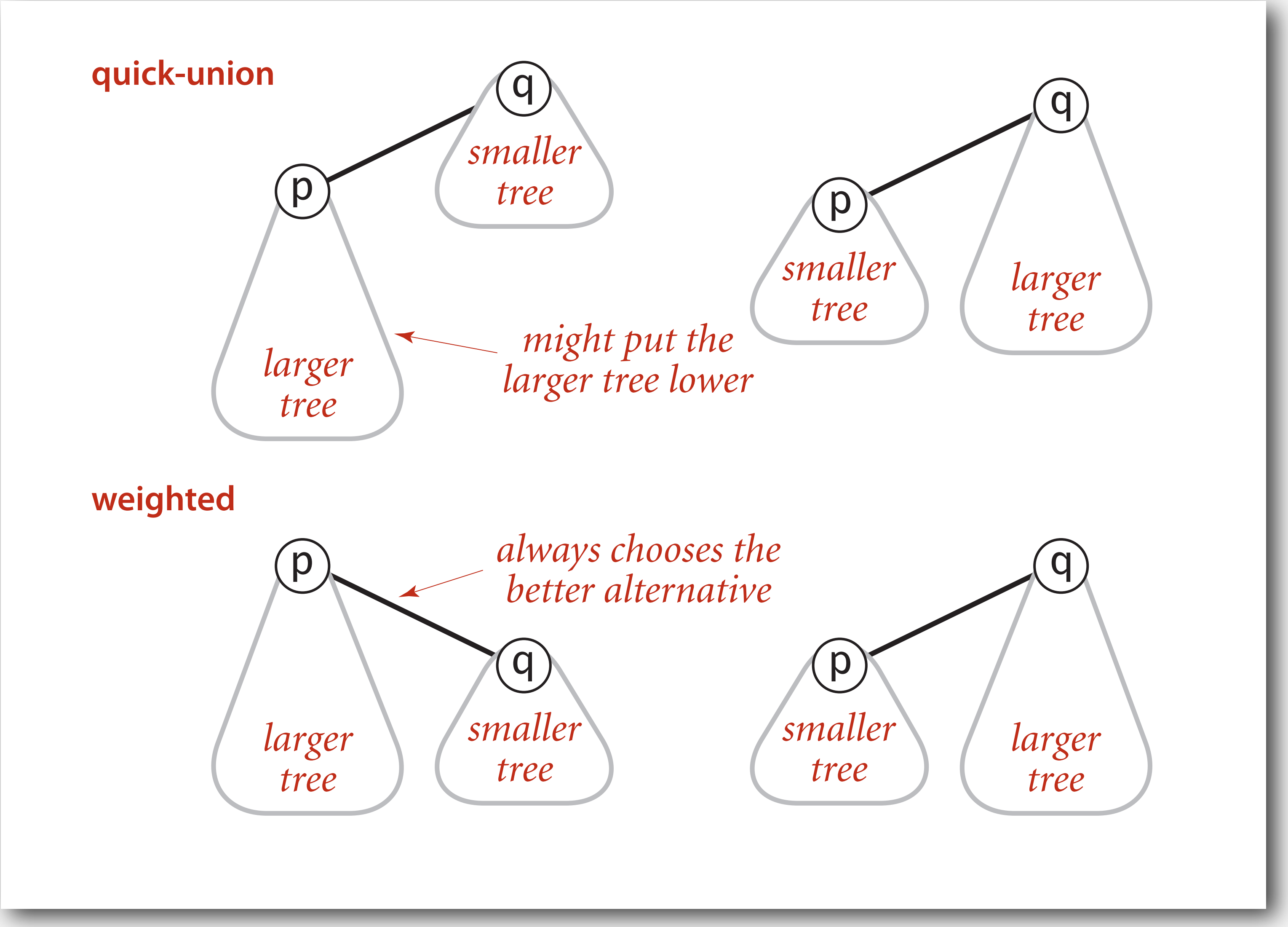
- **Quick-find defect.**
  - Union too expensive (N array accesses).
  - Trees are flat, but too expensive to keep them flat.

- **Quick-union defect.**
  - Trees can get tall.
  - Find too expensive (could be N array accesses).

- dynamic connectivity
- quick find
- quick union
- **improvements**
- applications

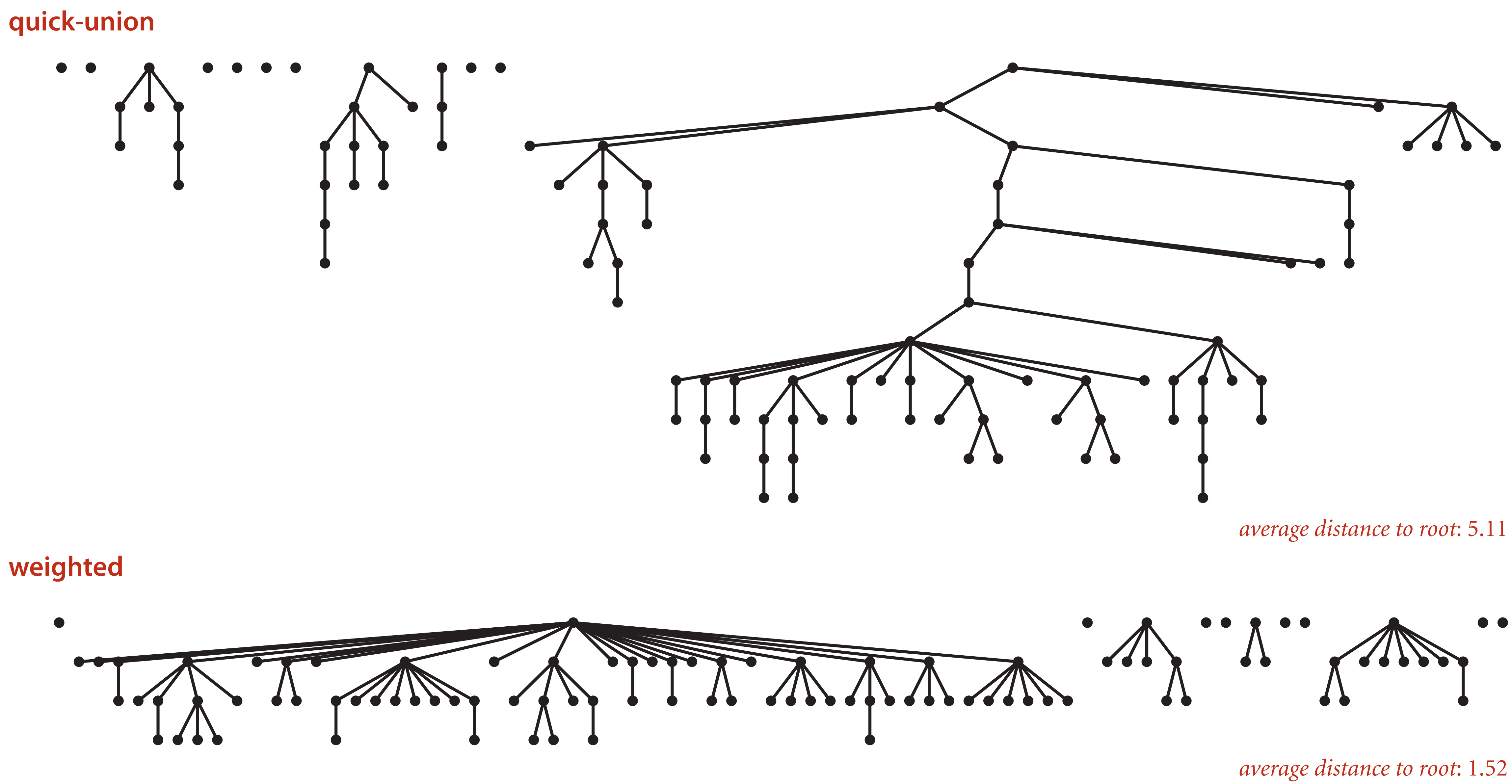
Improvement 1: weighting

- Weighted quick-union.
  - Modify quick-union to avoid tall trees.
  - Keep track of size of each tree (number of objects).
  - Balance by linking root of smaller tree to root of larger tree.



Weighted quick-union demo

Quick-union and weighted quick-union example



Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

- Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

- Find. Identical to quick-union.

```
return root(p) == root(q);
```

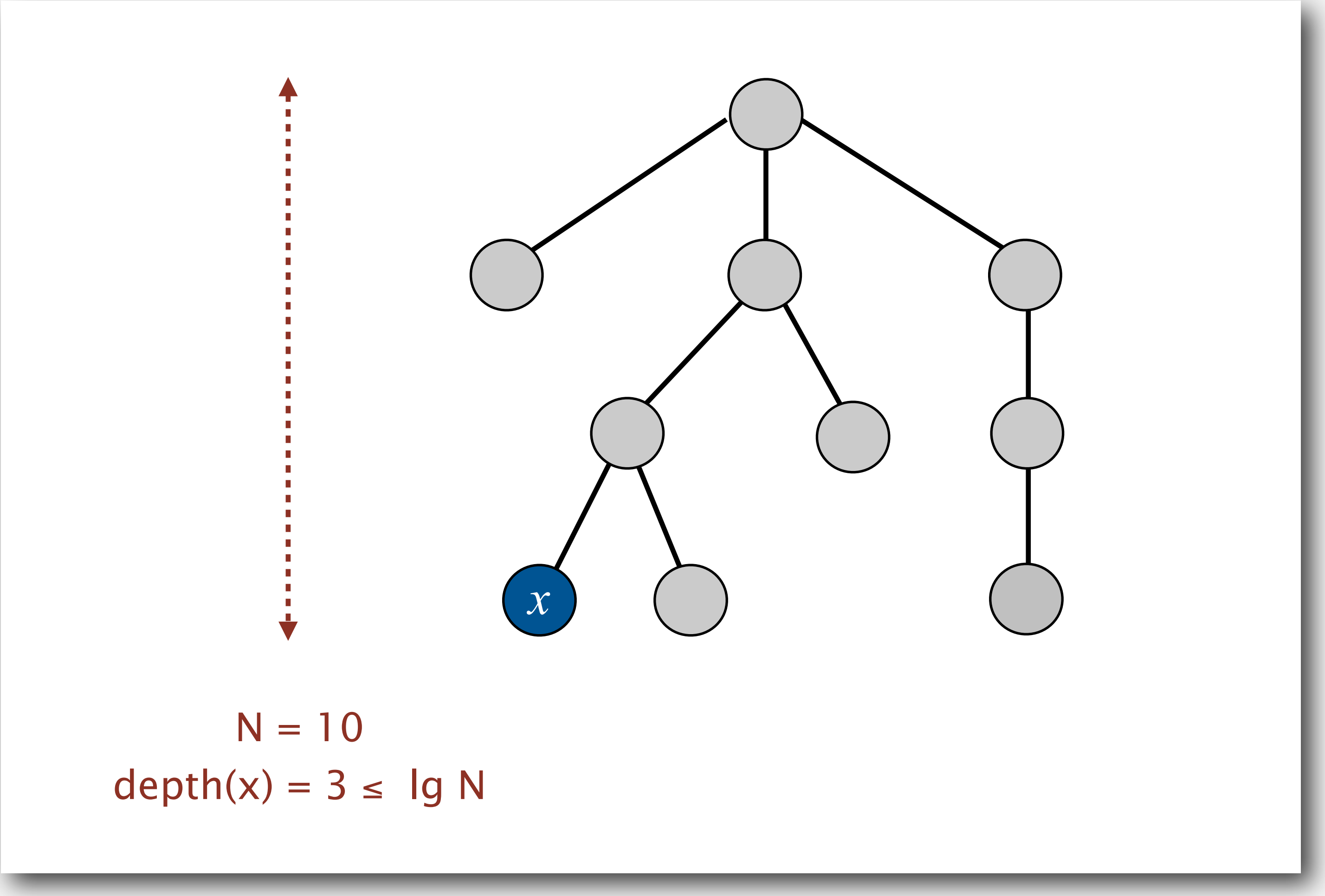
- Union. Modify quick-union to:
  - Link root of smaller tree to root of larger tree.
  - Update the `sz[]` array.

```
int i = root(p);
int j = root(q);
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

Weighted quick-union analysis

- Running time.
  - Find: takes time proportional to depth of p and q.
  - Union: takes constant time, given roots.

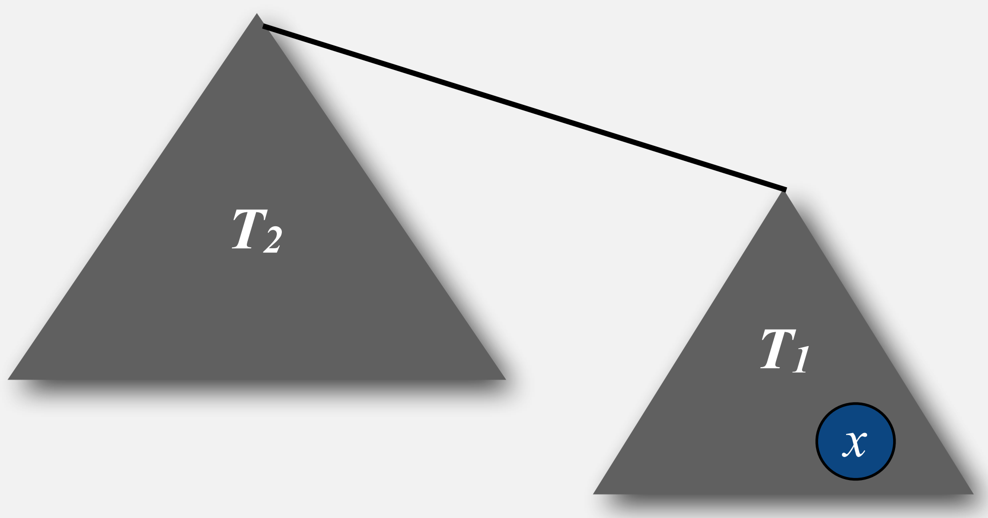
- Proposition. Depth of any node x is at most $\lg N$.



Weighted quick-union analysis

- Running time.
  - Find: takes time proportional to depth of p and q.
  - Union: takes constant time, given roots.

- Proposition. Depth of any node x is at most $\lg N$.
- Pf. When does depth of x increase?
- Increases by 1 when tree T_1 containing x is merged into another tree T_2.
  - The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
  - Size of tree containing x can double at most $\lg N$ times. Why?



Weighted quick-union analysis

- Running time.
  - Find: takes time proportional to depth of p and q.
  - Union: takes constant time, given roots.

- Proposition. Depth of any node x is at most $\lg N$.

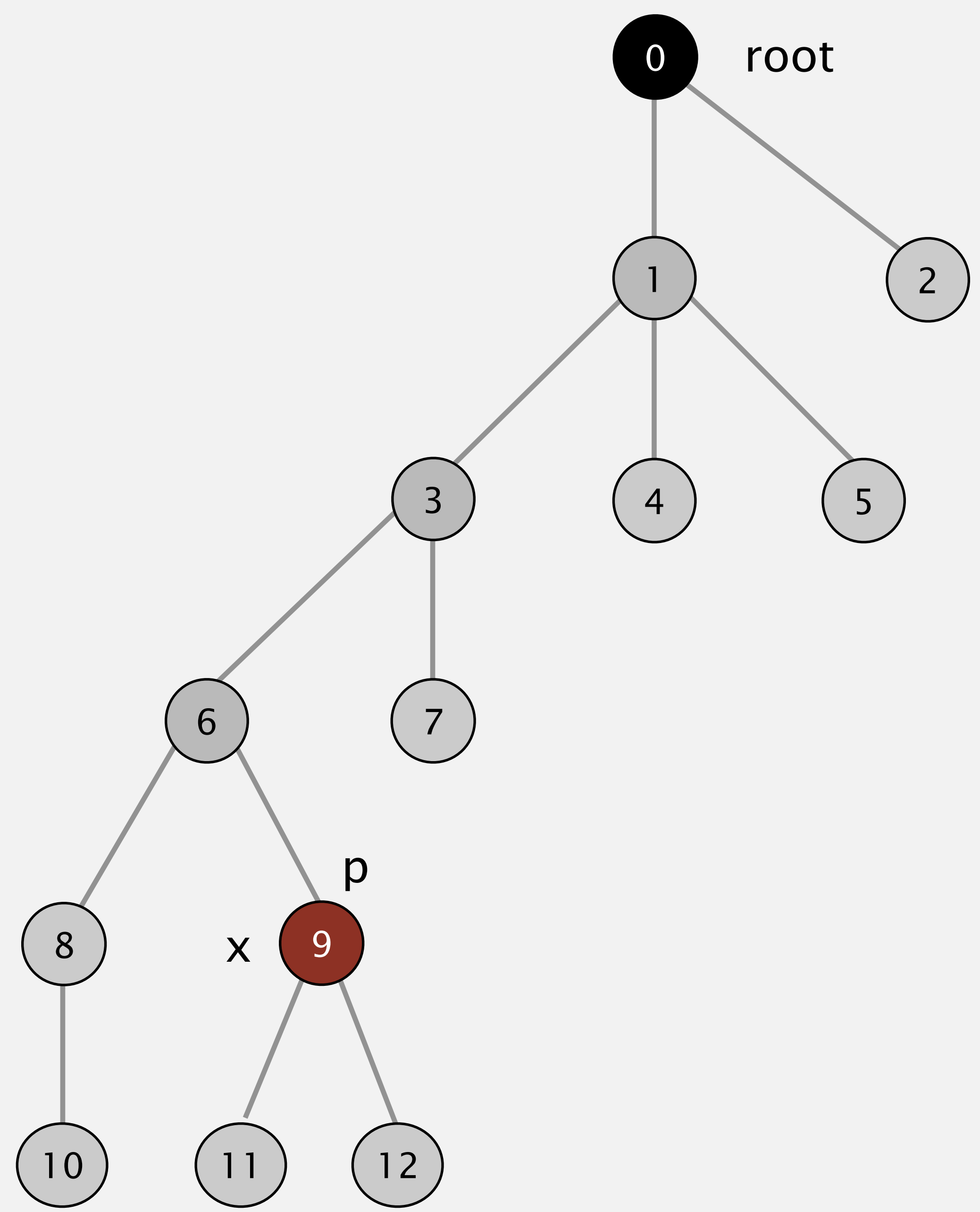
algorithm	initialize	union	connected
quick-find	N	N	1
quick-union	N	N †	N
weighted QU	N	lg N †	lg N

† includes cost of finding roots

- Q. Stop at guaranteed acceptable performance?
- A. No, easy to improve further.

Improvement 2: path compression

- Quick union with path compression. Just after computing the root of p,
- set the id of each examined node to point to that root.



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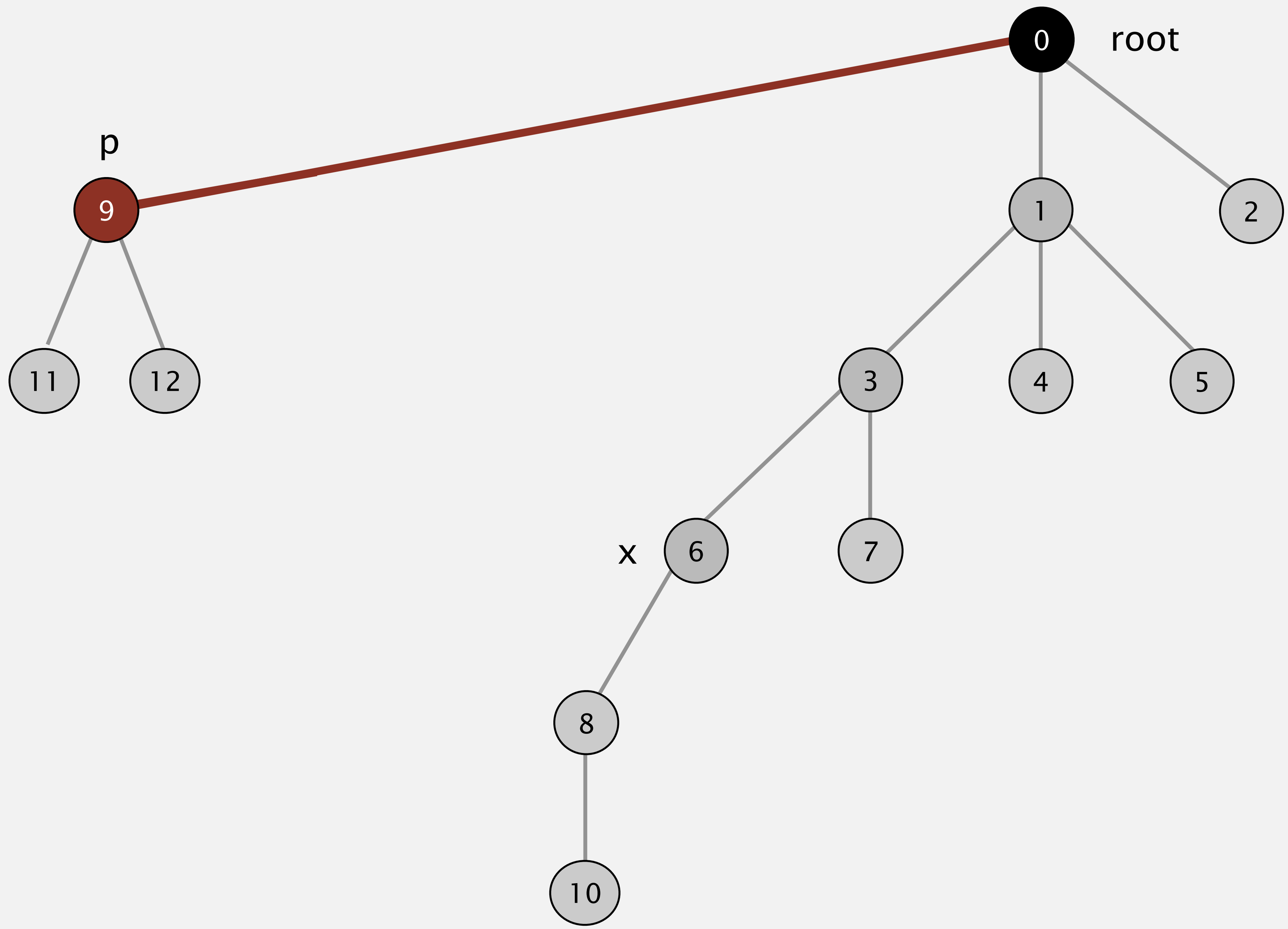

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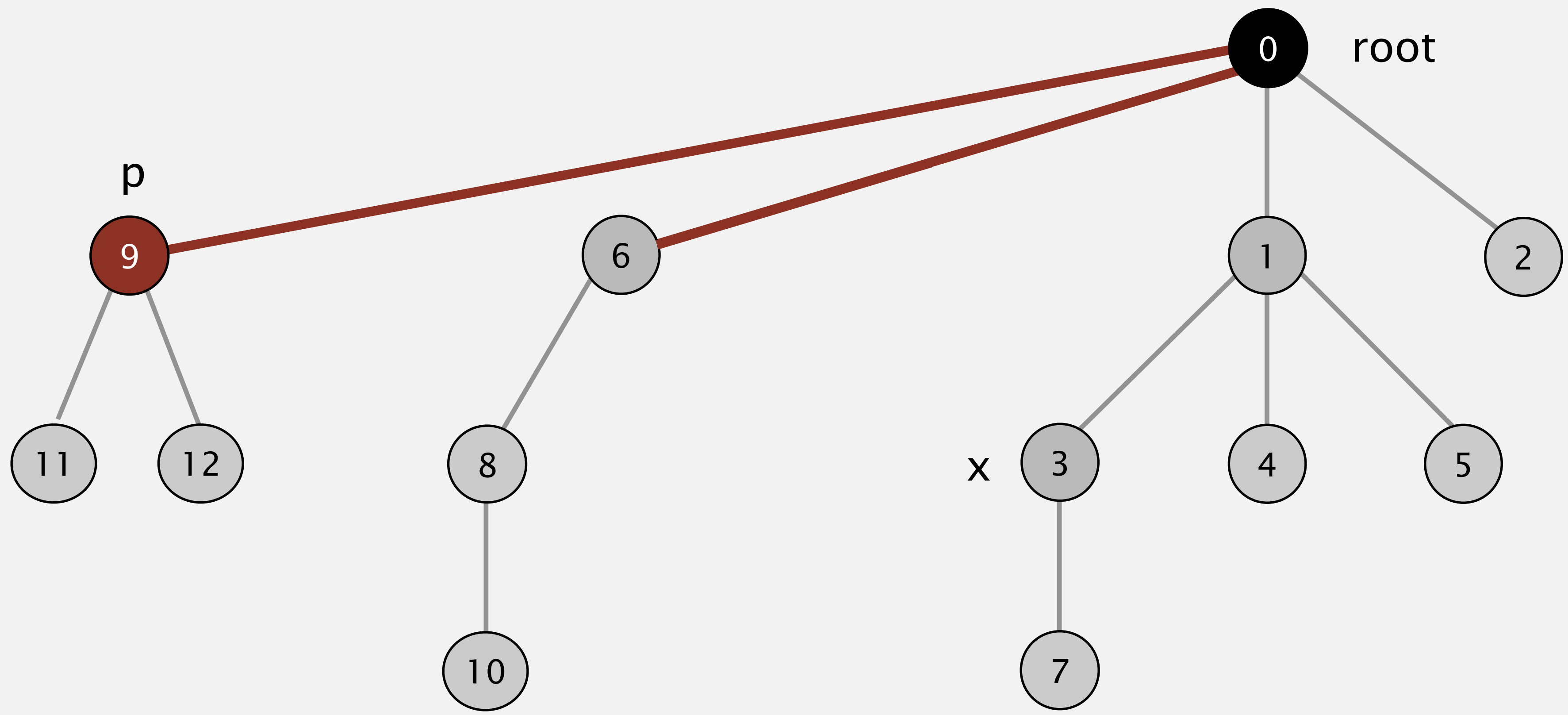

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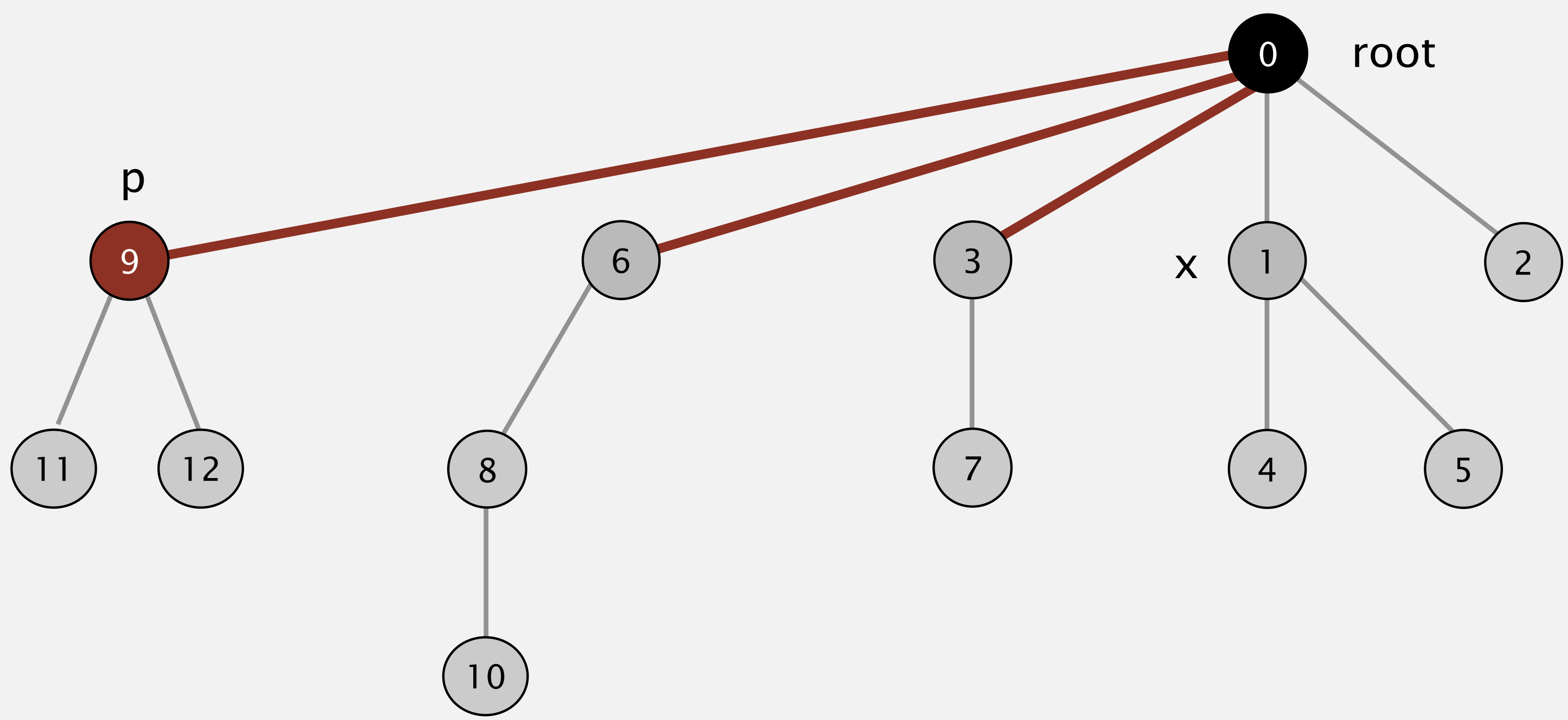

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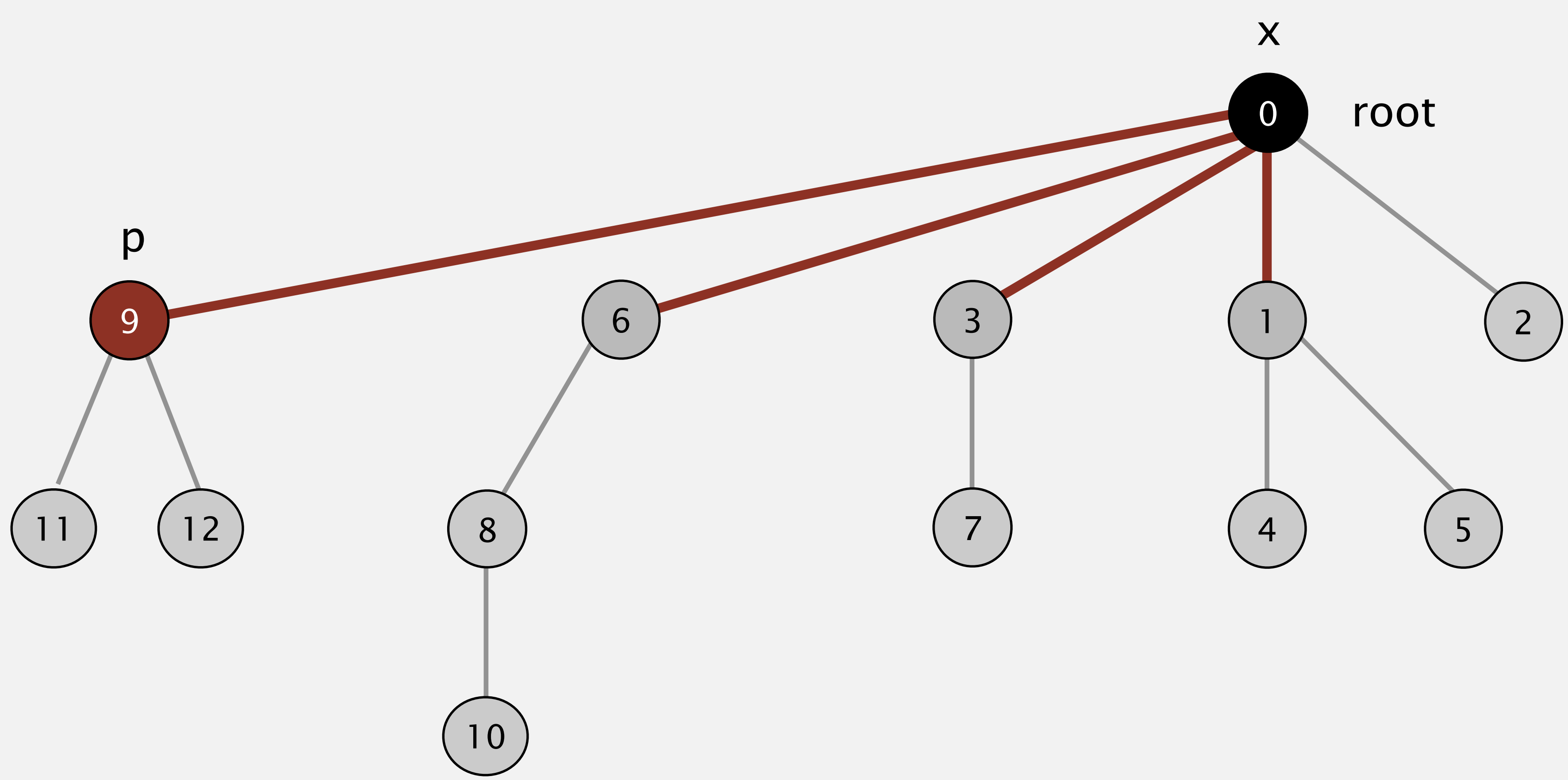


Path compression: Java implementation

- Two-pass implementation: add second loop to `root()` to set the `id[]` of each examined node to the root.

- Simpler one-pass variant: Make every other node in path point to its grandparent (thereby halving path length).

```
private int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

- In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression: amortized analysis

- **Proposition.** Starting from an empty data structure,
- any sequence of M union–find operations on N objects makes at most proportional to $N + M \lg^* N$ array accesses.
  - Proof is very difficult.
  - But the algorithm is simple!
  - Analysis can be improved to $N + M\, \alpha(M, N)$.

see COS 423

Bob Tarjan
(Turing Award '86)

- **Linear-time algorithm for M union-find ops on N objects?**
  - Cost within constant factor of reading in the data.
  - In theory, WQUPC is not quite linear.
  - In practice, WQUPC is linear.

because $\lg^* N$ is a constant in this universe

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

lg* function

- **Amazing fact.** No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

- **Bottom line.** Weighted quick union (with path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

M union-find operations on a set of N objects

- **Ex.** [10^9 unions and finds with 10^9 objects]
  - WQUPC reduces time from 30 years to 6 seconds.
  - Supercomputer won't help much; good algorithm enables solution.

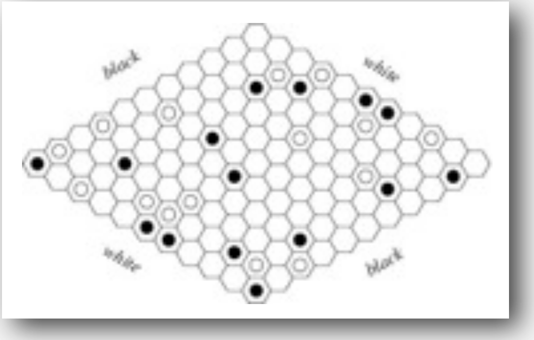
- dynamic connectivity
- quick find
- quick union
- improvements
- **applications**

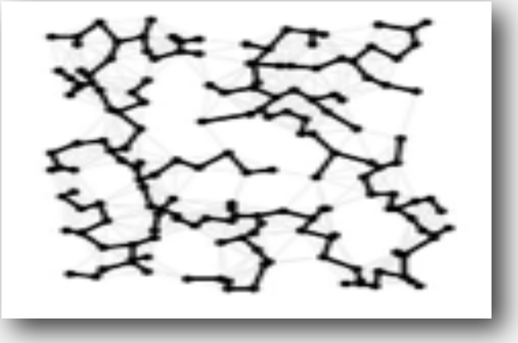
Union-find applications

- Percolation. ← see also Assignment 1
- Games (Go, Hex).

✓ Dynamic connectivity.

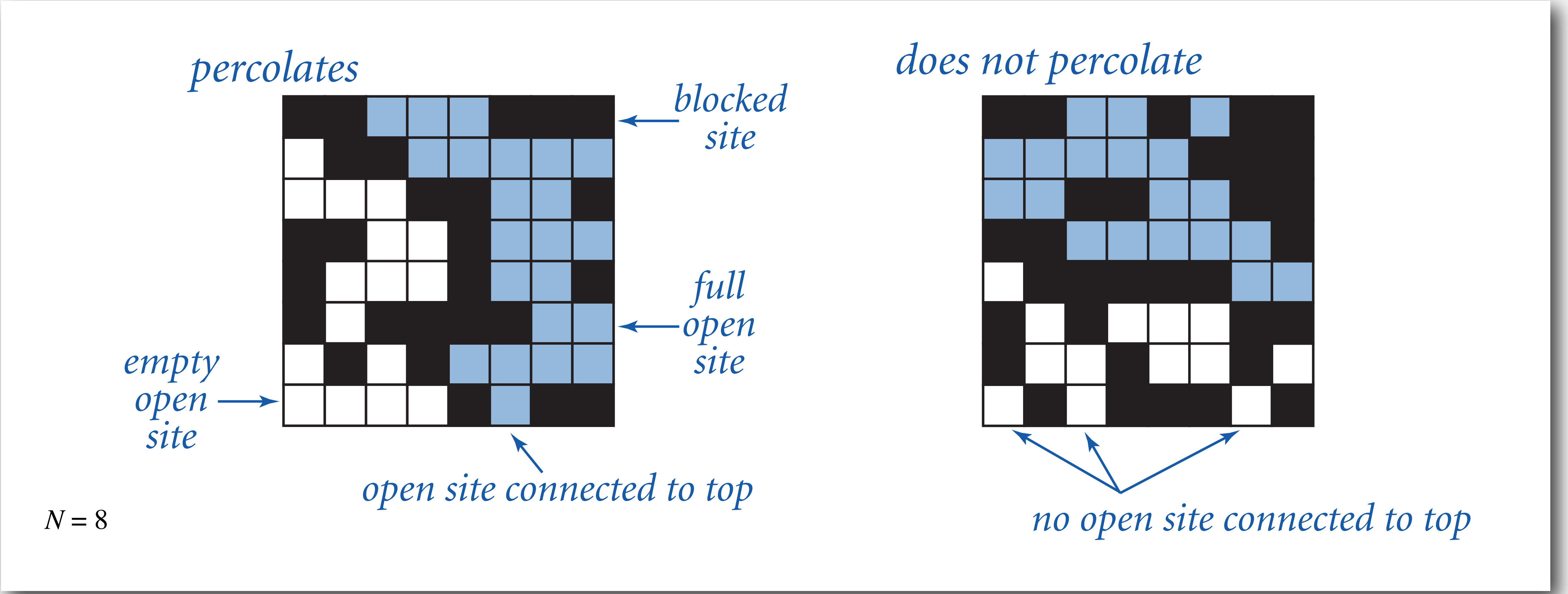
- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's `bwlabel()` function in image processing.





Percolation

- A model for many physical systems:
  - N-by-N grid of sites.
  - Each site is open with probability p (or blocked with probability $1 - p$).
  - System percolates if and only if top and bottom are connected by open sites.



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  - N-by-N grid of sites.
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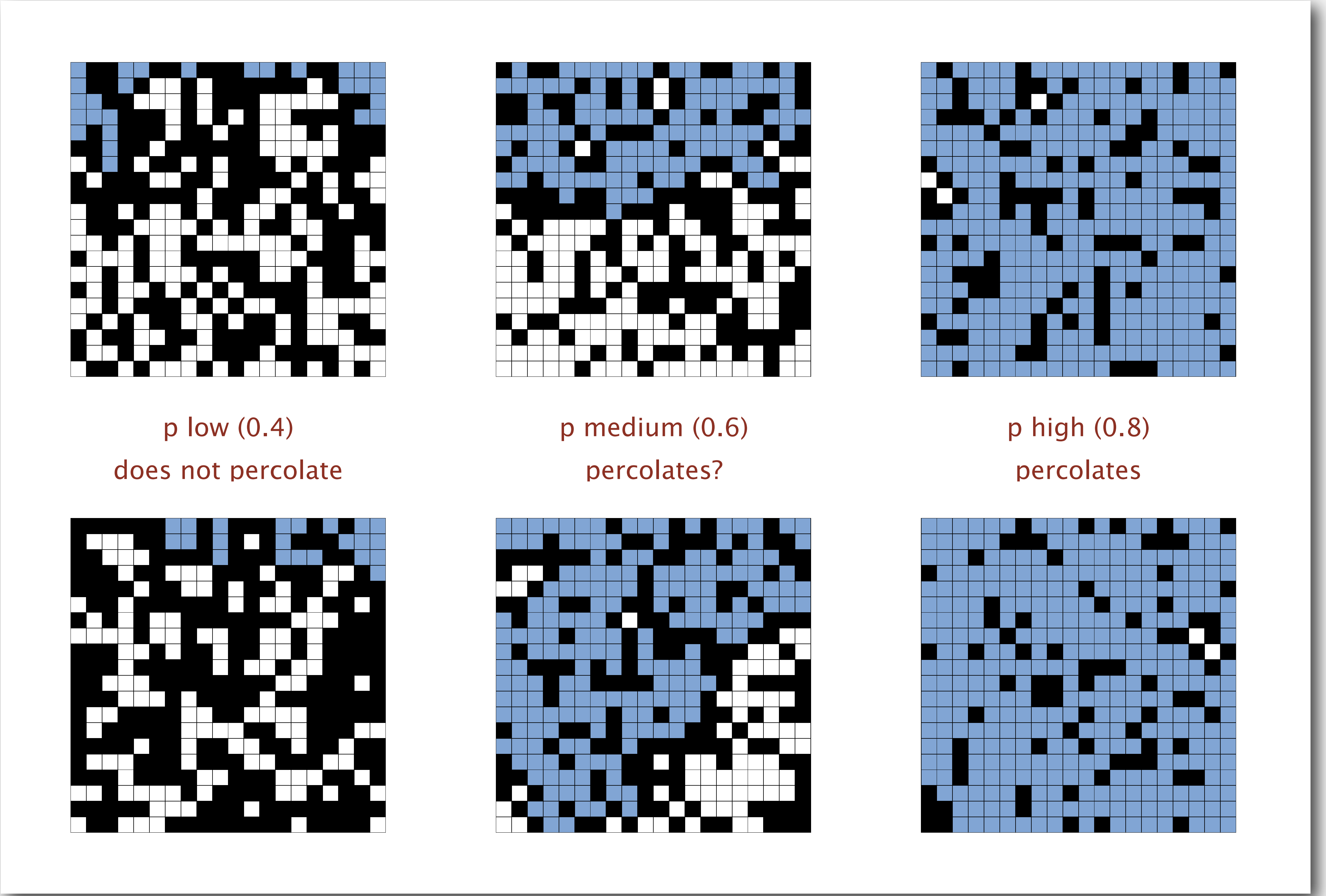
  - System percolates if and only if top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

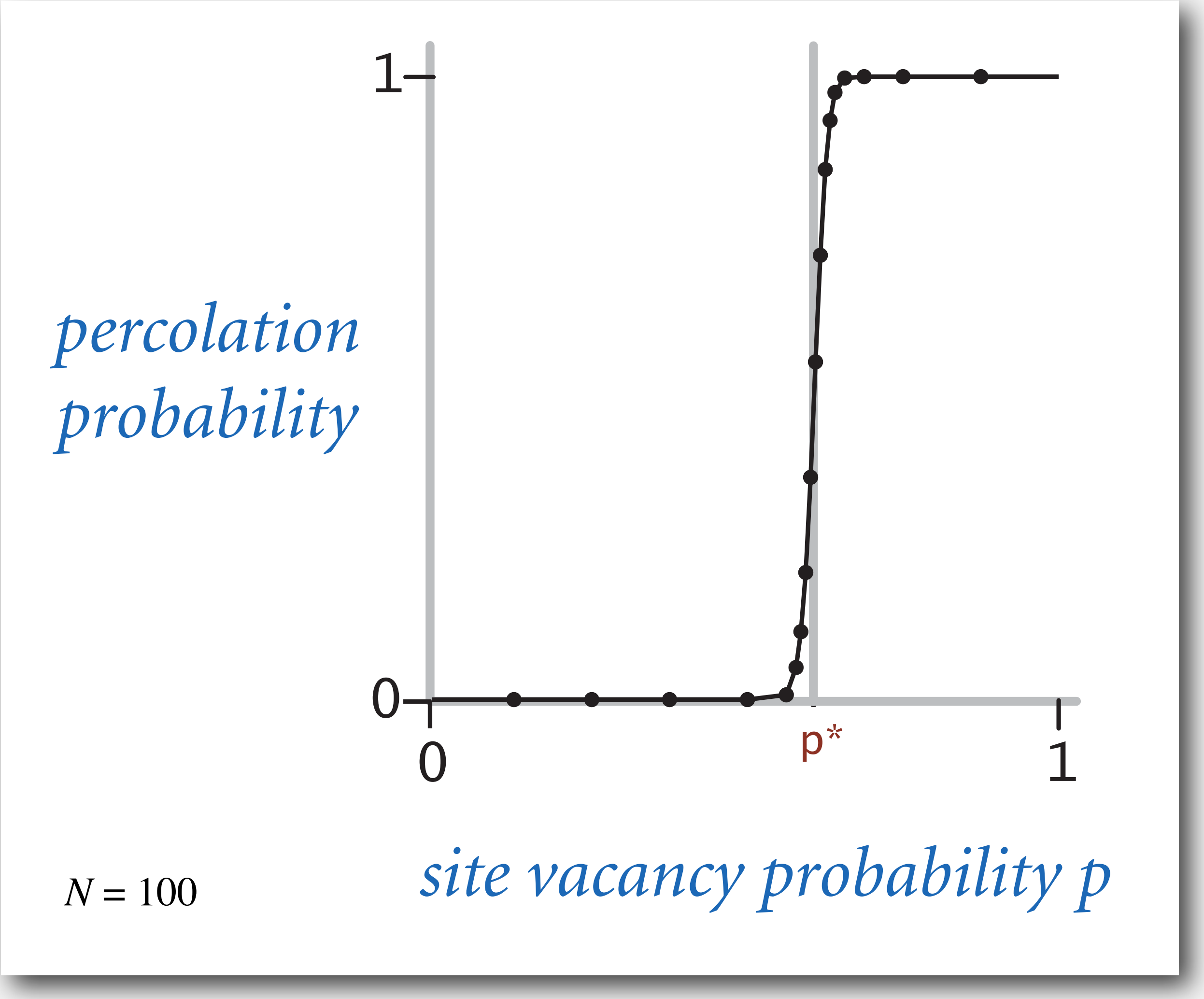
- Depends on site vacancy probability p.



Percolation phase transition

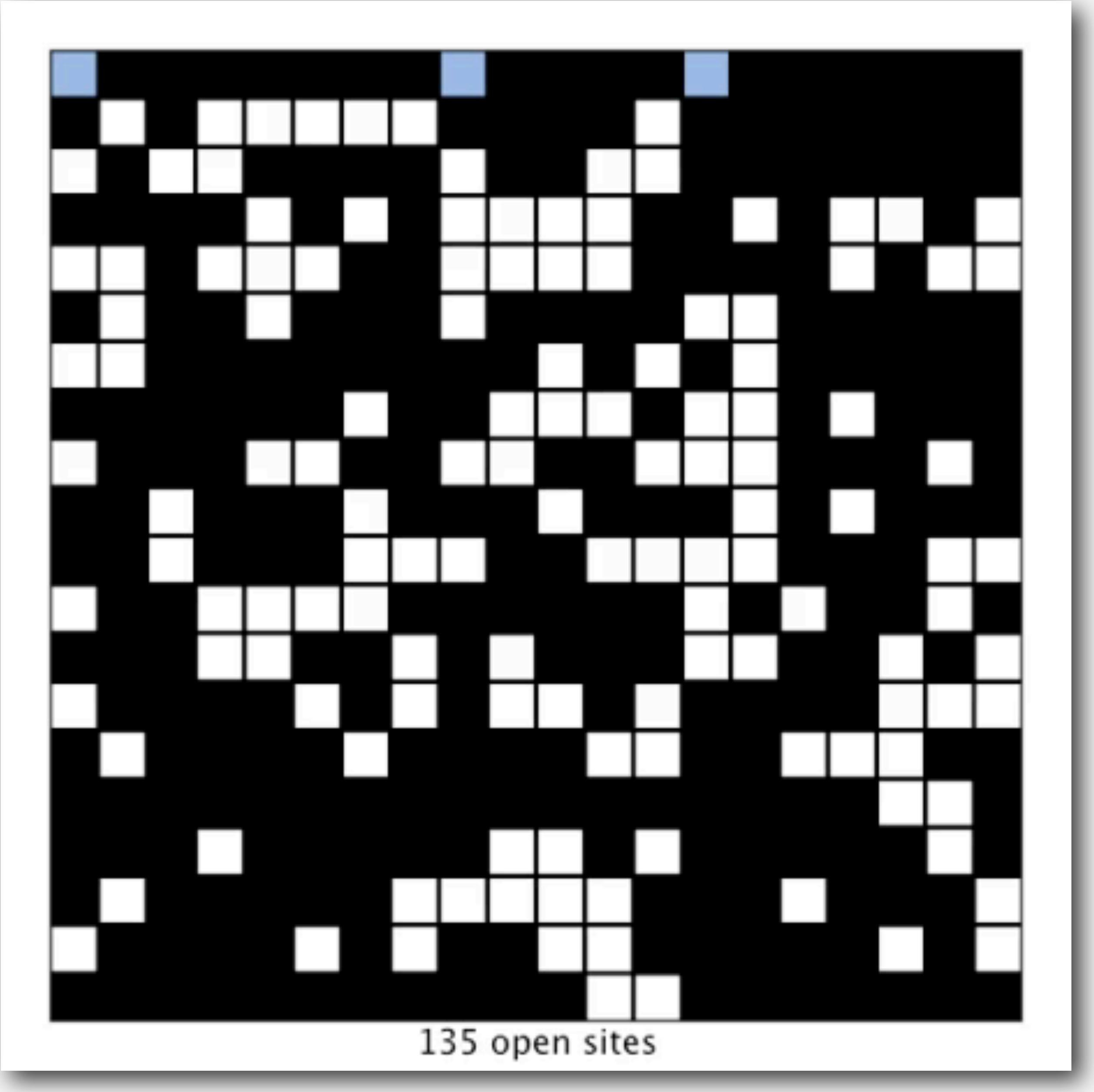
- When N is large, theory guarantees a sharp threshold p^*.
  - $p > p^*$: almost certainly percolates.
  - $p < p^*$: almost certainly does not percolate.

- Q. What is the value of p^* ?



Monte Carlo simulation

- Initialize N-by-N whole grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.



$N = 20$

full open site (connected to top)

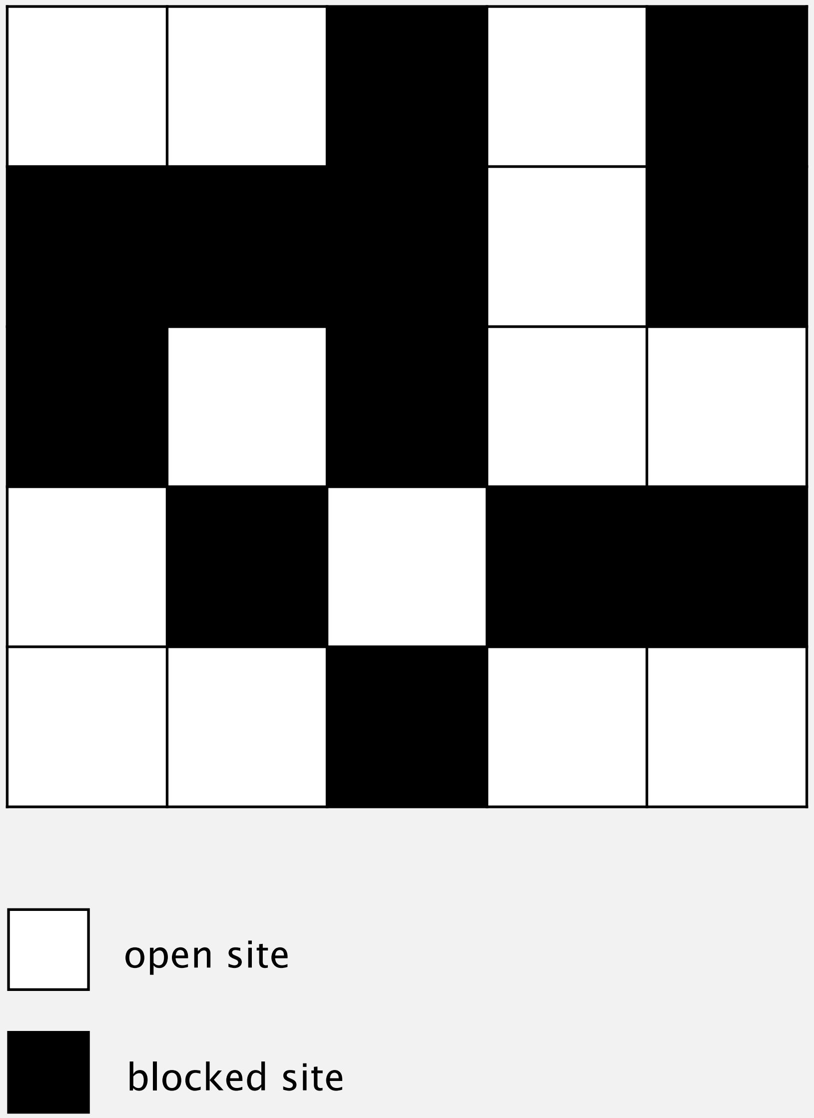
empty open site (not connected to top)

blocked site

Dynamic connectivity solution to estimate percolation threshold

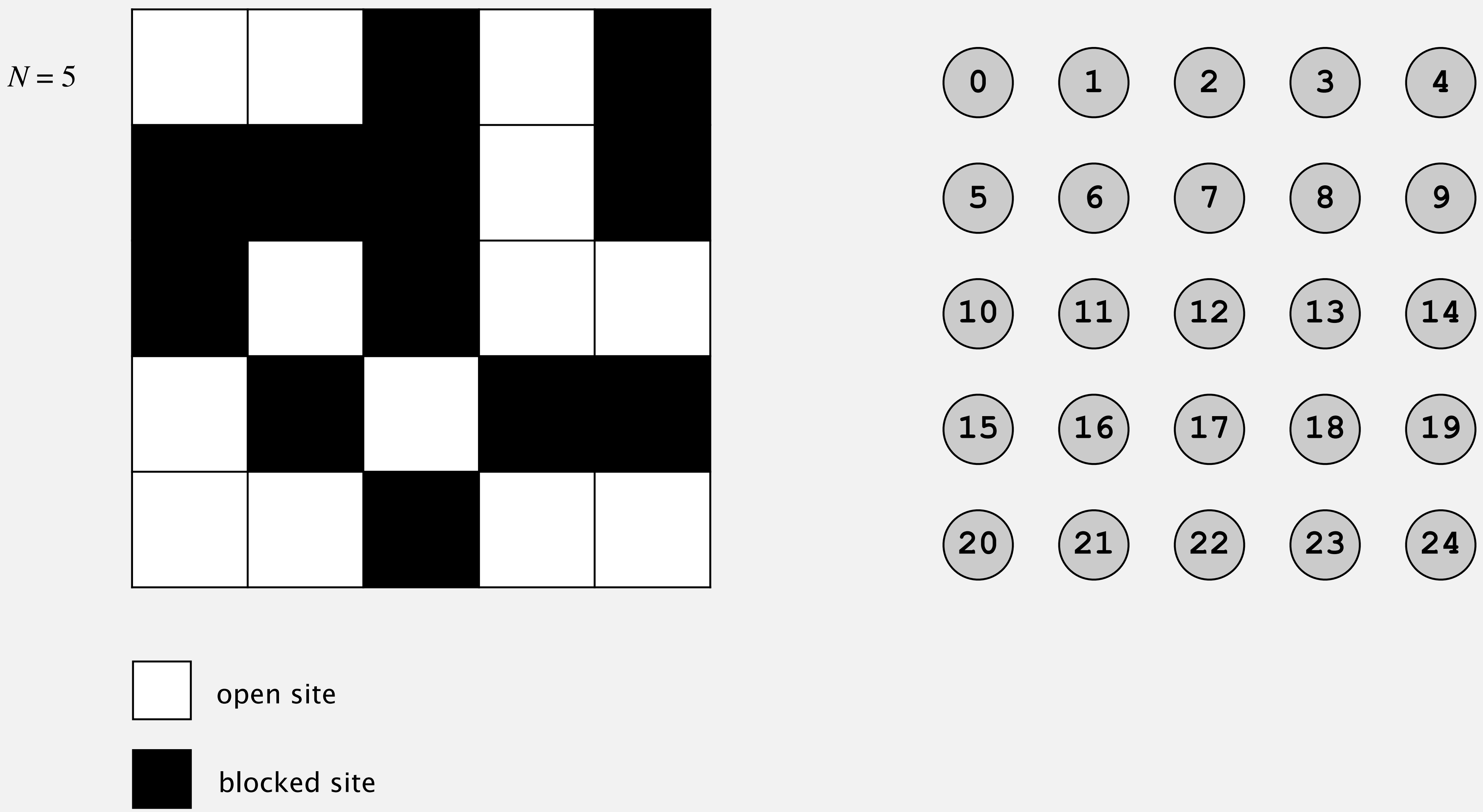
- Q. How to check whether an N-by-N system percolates?

$N = 5$



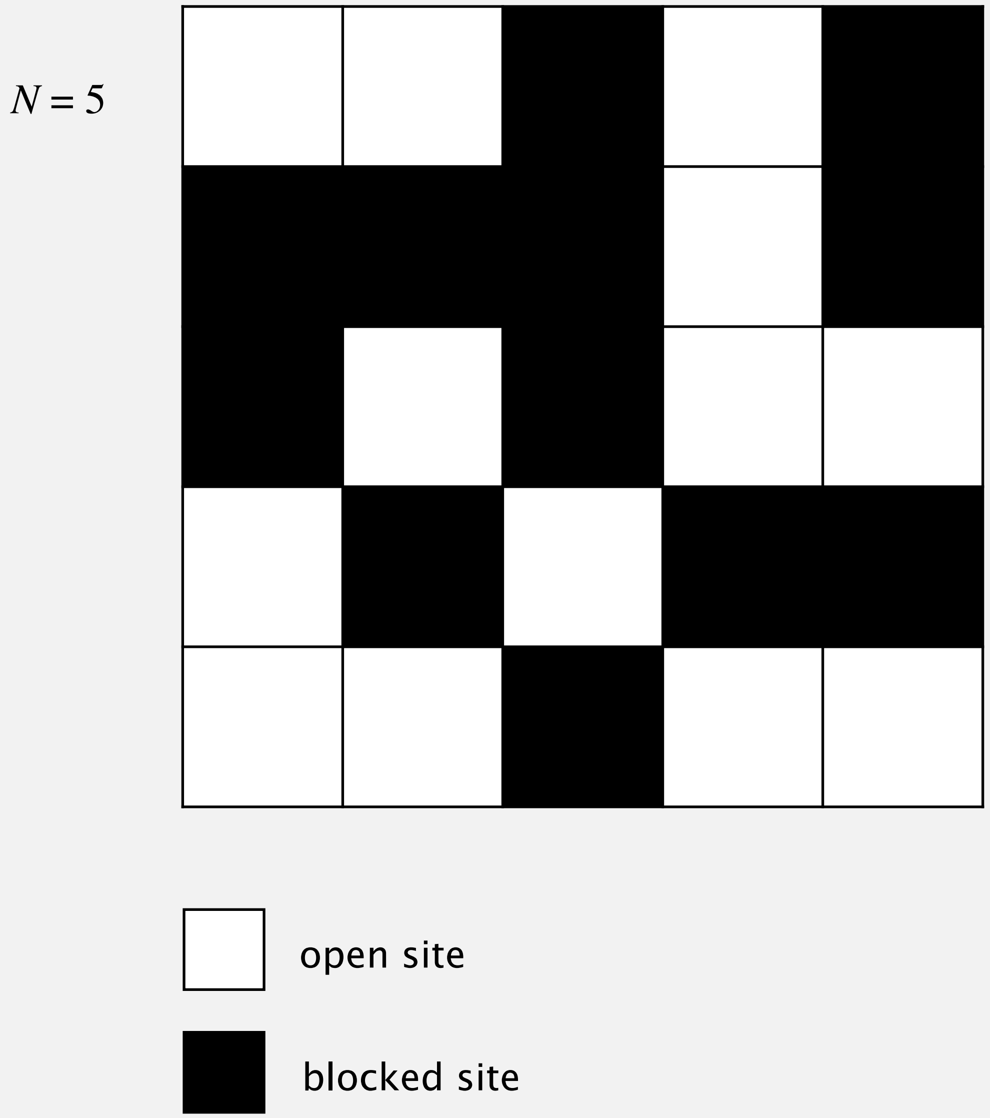
Dynamic connectivity solution to estimate percolation threshold

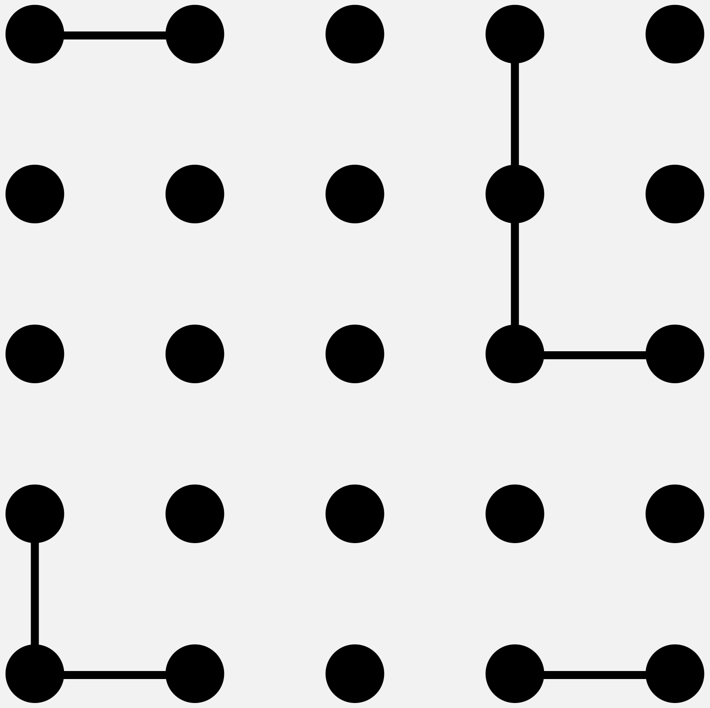
- Q. How to check whether an N-by-N system percolates?
  - Create an object for each site and name them 0 to $N^2 - 1$.



Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an N-by-N system percolates?
  - Create an object for each site and name them 0 to $N^2 - 1$.
  - Sites are in same component if connected by open sites.



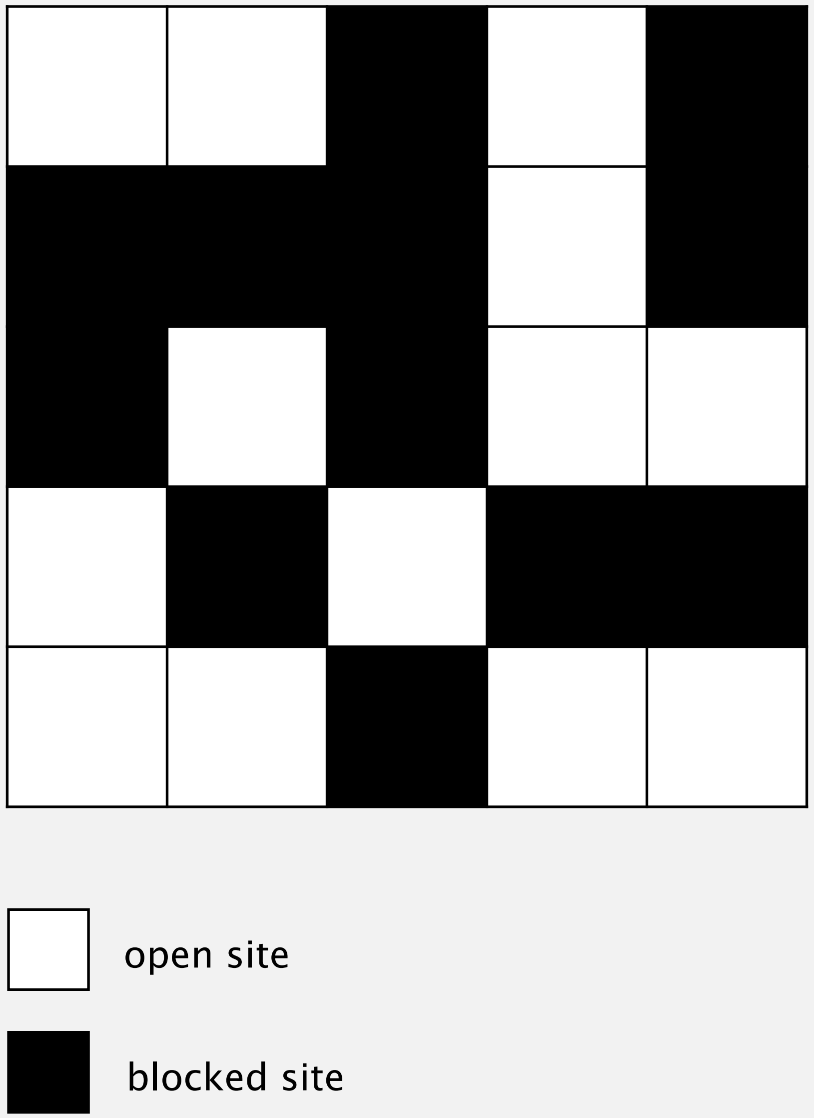


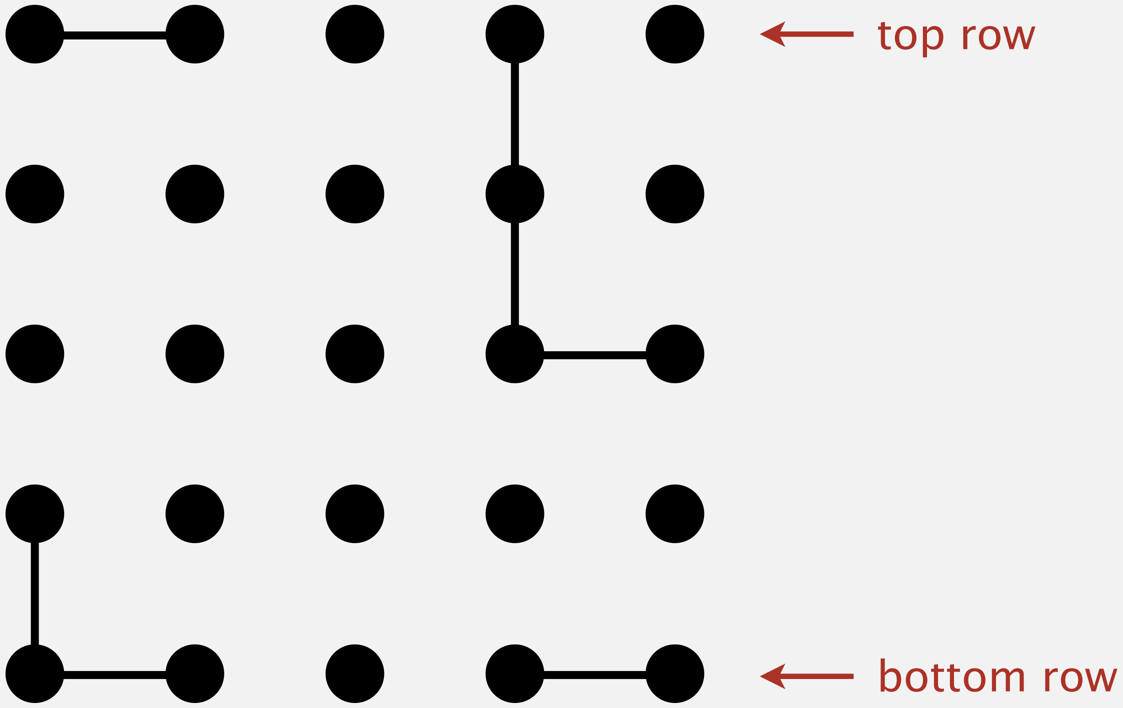
Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an N-by-N system percolates?
  - Create an object for each site and name them 0 to $N^2 - 1$.
  - Sites are in same component if connected by open sites.
  - Percolates iff any site on bottom row is connected to site on top row.

brute-force algorithm: N^2 calls to connected()

$N = 5$

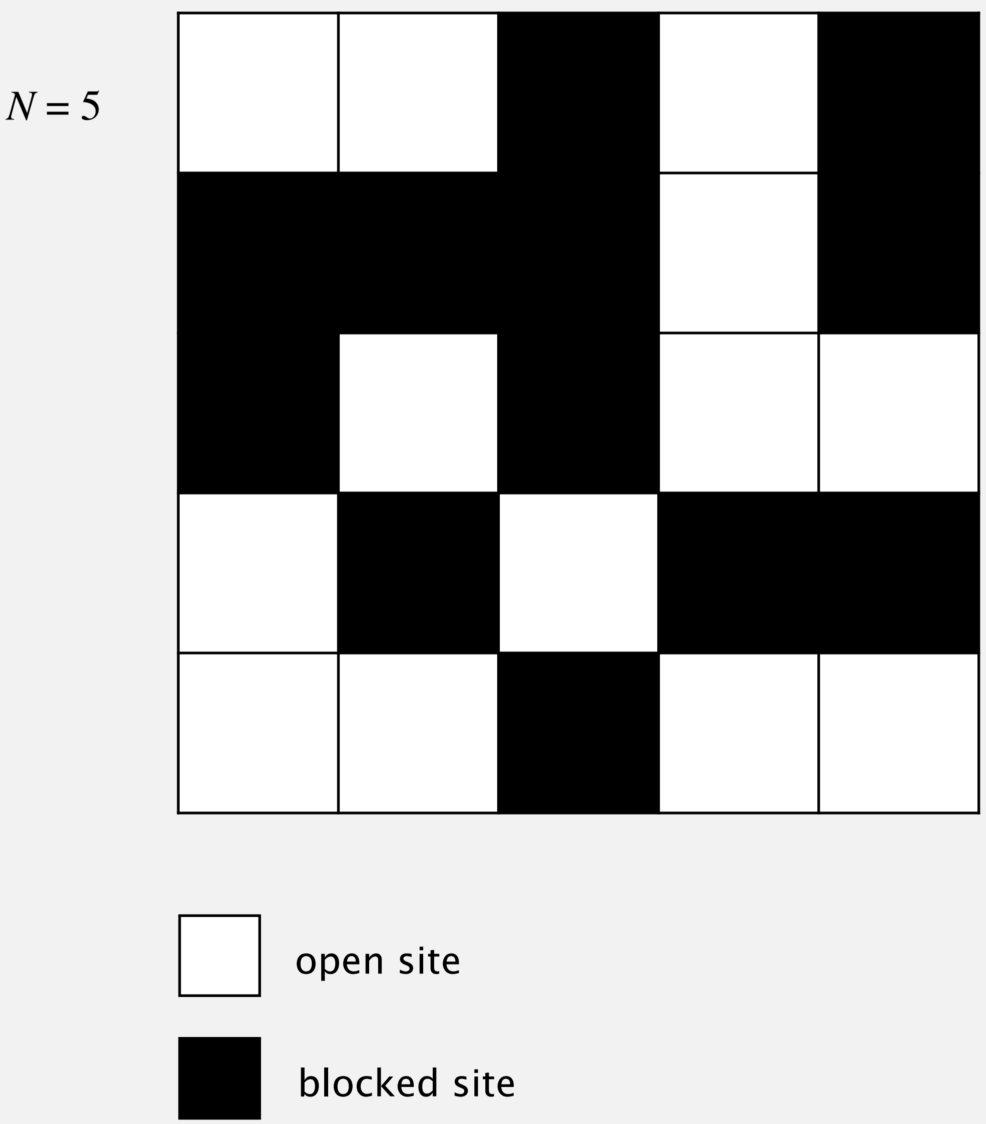


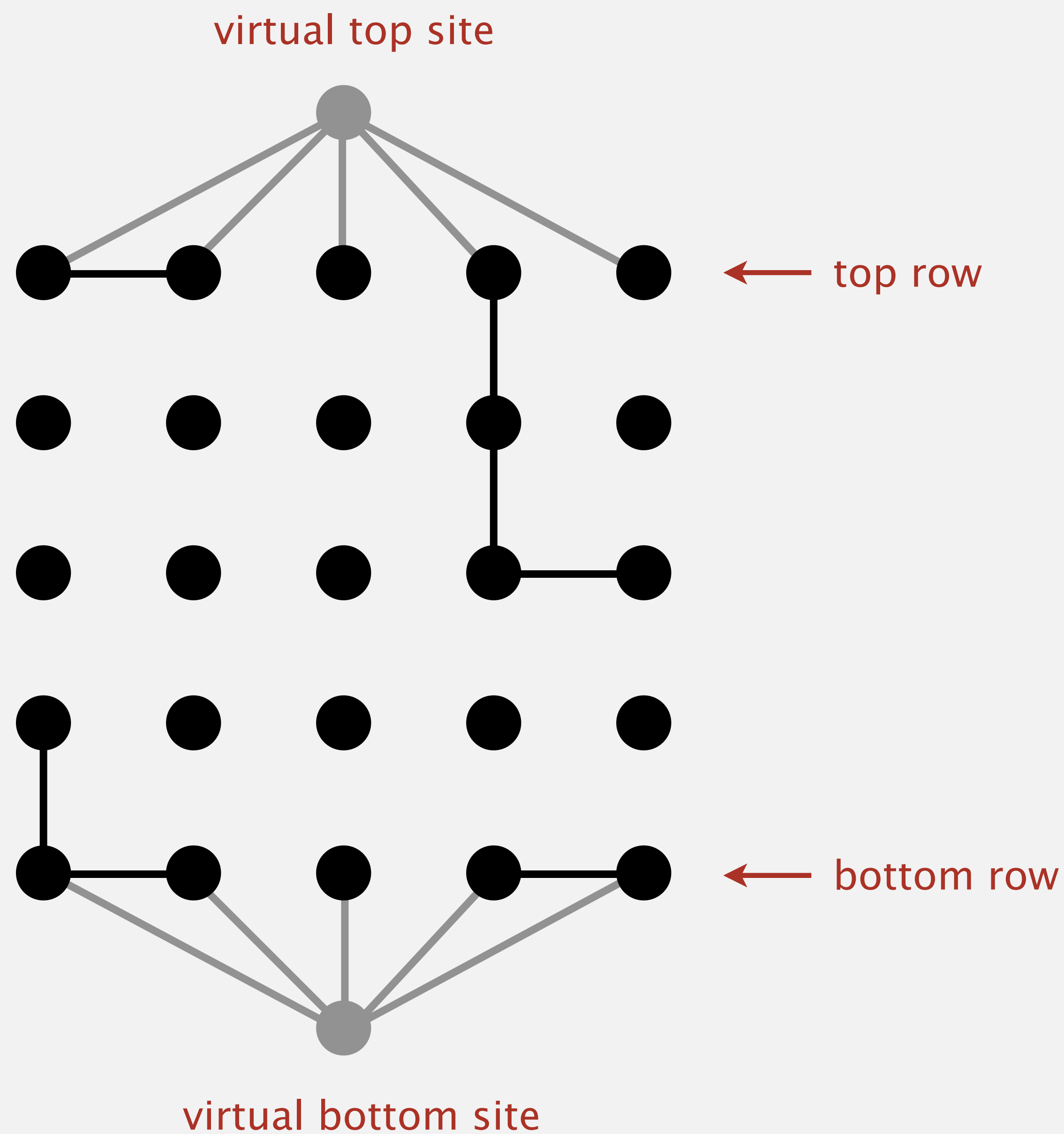


Dynamic connectivity solution to estimate percolation threshold

- **Clever trick.** Introduce two virtual sites (and connections to top and bottom).
  - Percolates iff virtual top site is connected to virtual bottom site.

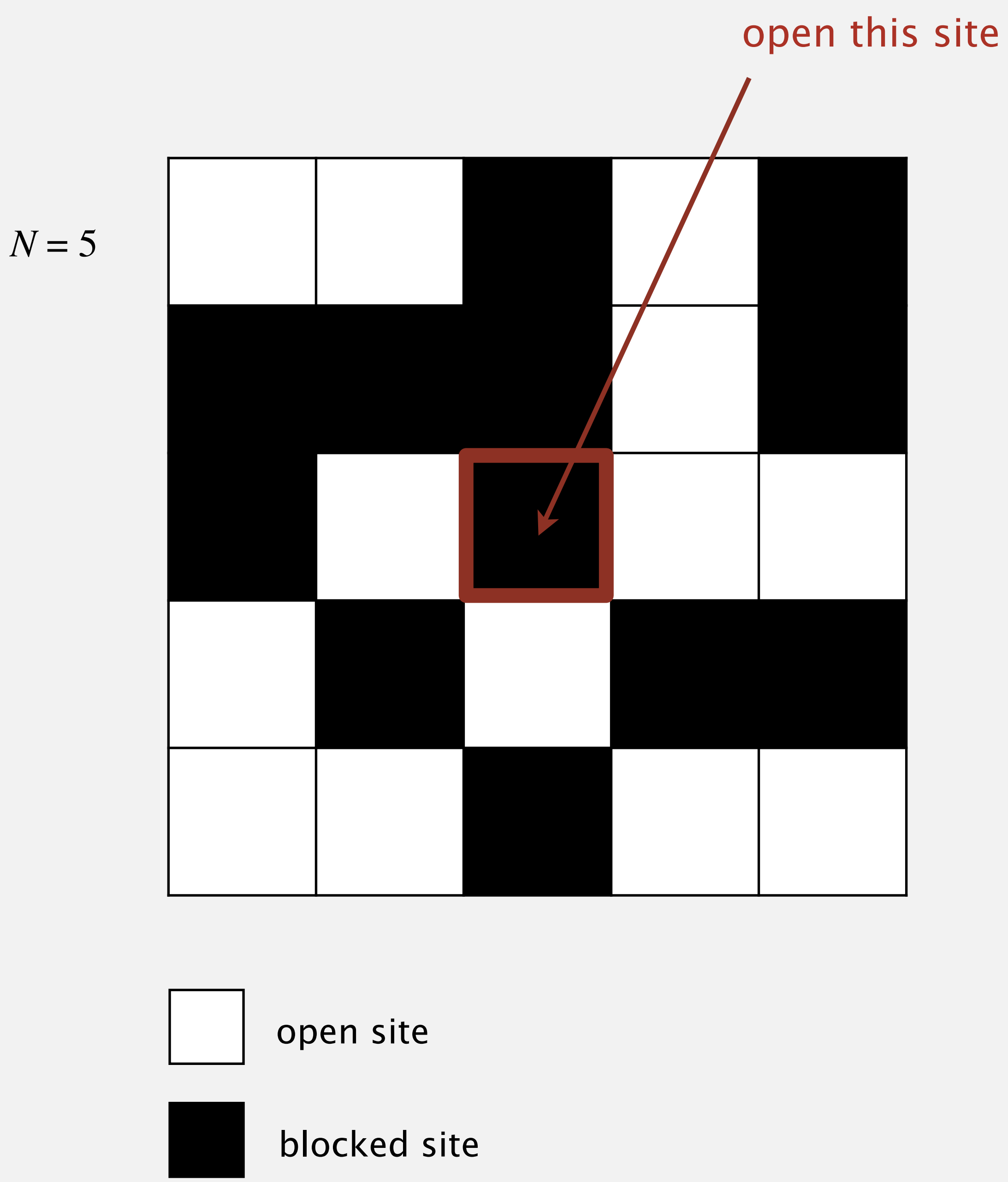
efficient algorithm: only 1 call to connected()

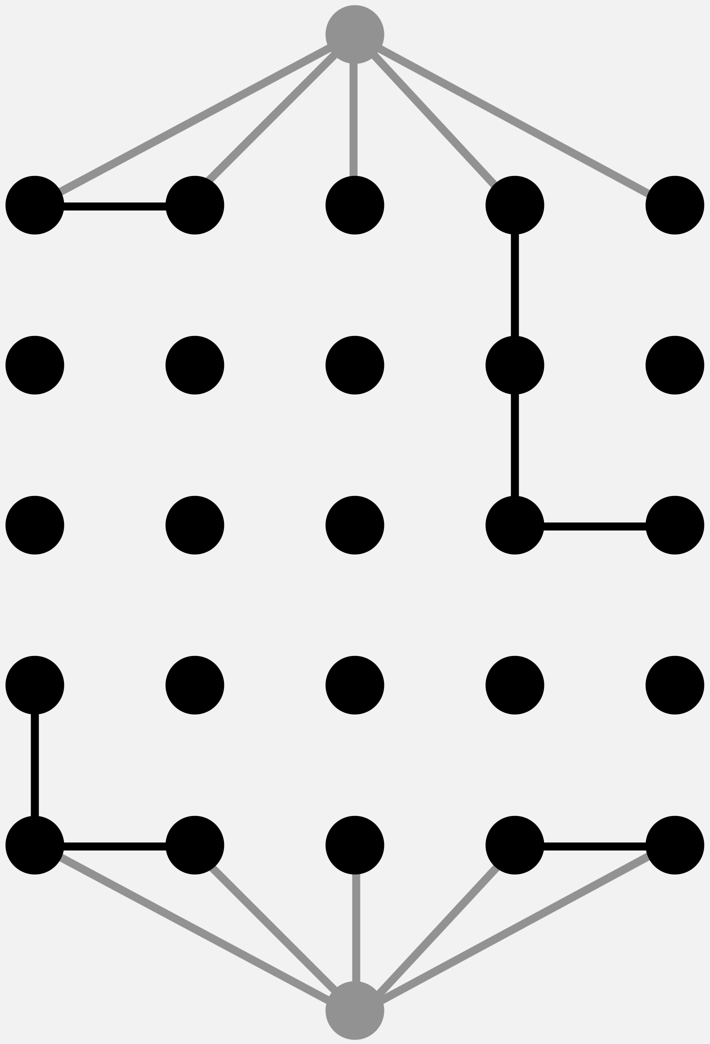




Dynamic connectivity solution to estimate percolation threshold

- **Q.** How to model as dynamic connectivity problem when opening a new site?





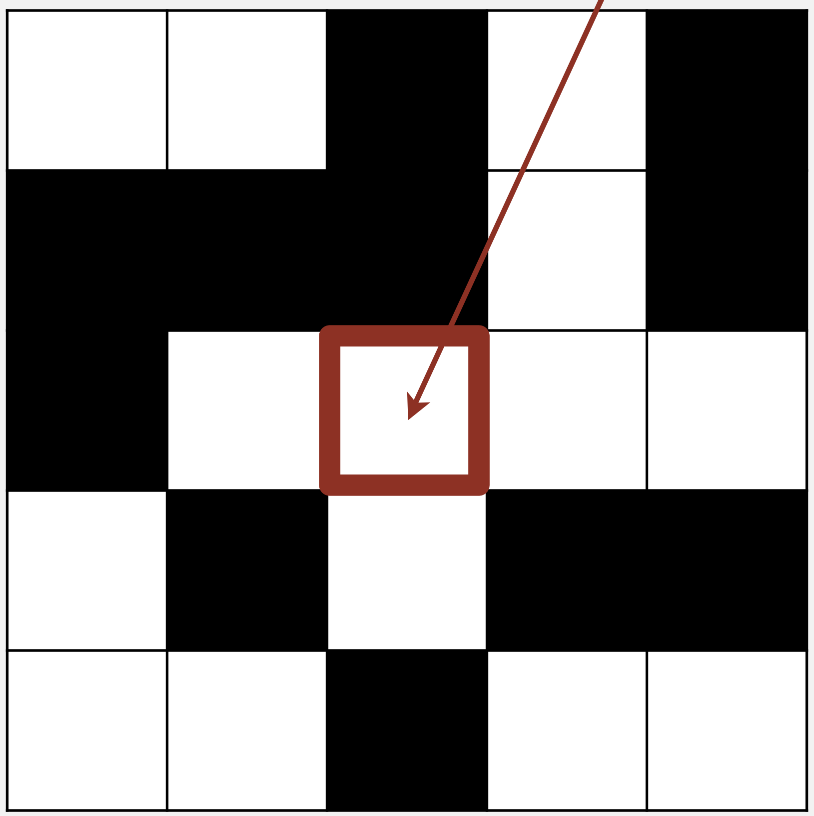
Dynamic connectivity solution to estimate percolation threshold

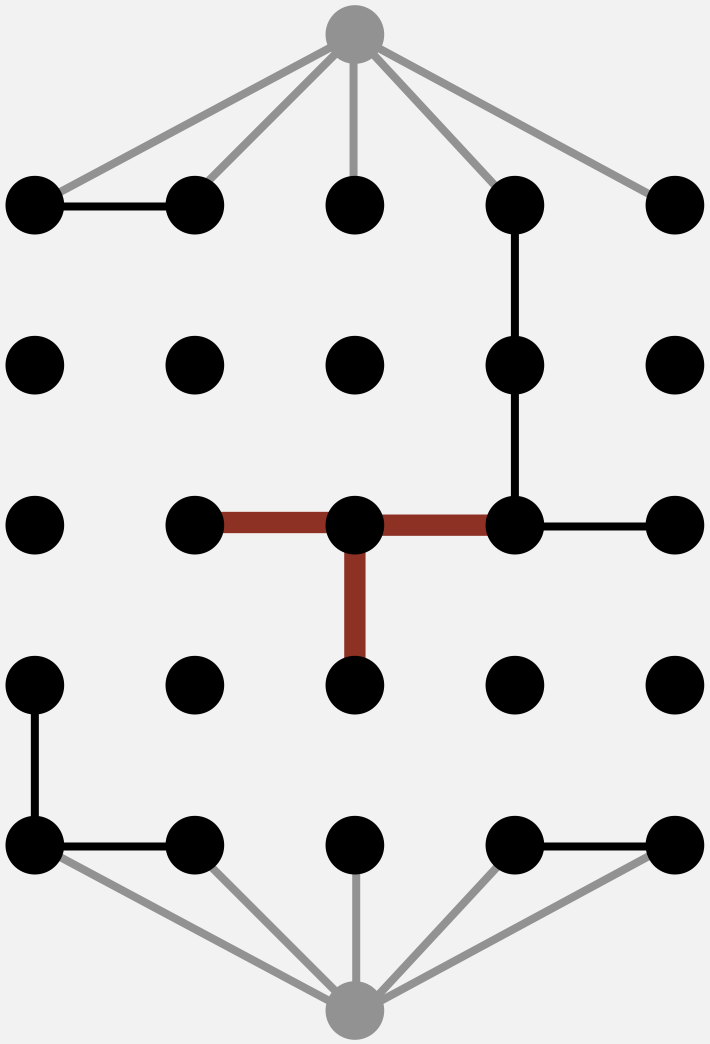
- **Q.** How to model as dynamic connectivity problem when opening a new site?
- **A.** Connect newly opened site to all of its adjacent open sites.

up to 4 calls to union()

open this site

$N = 5$





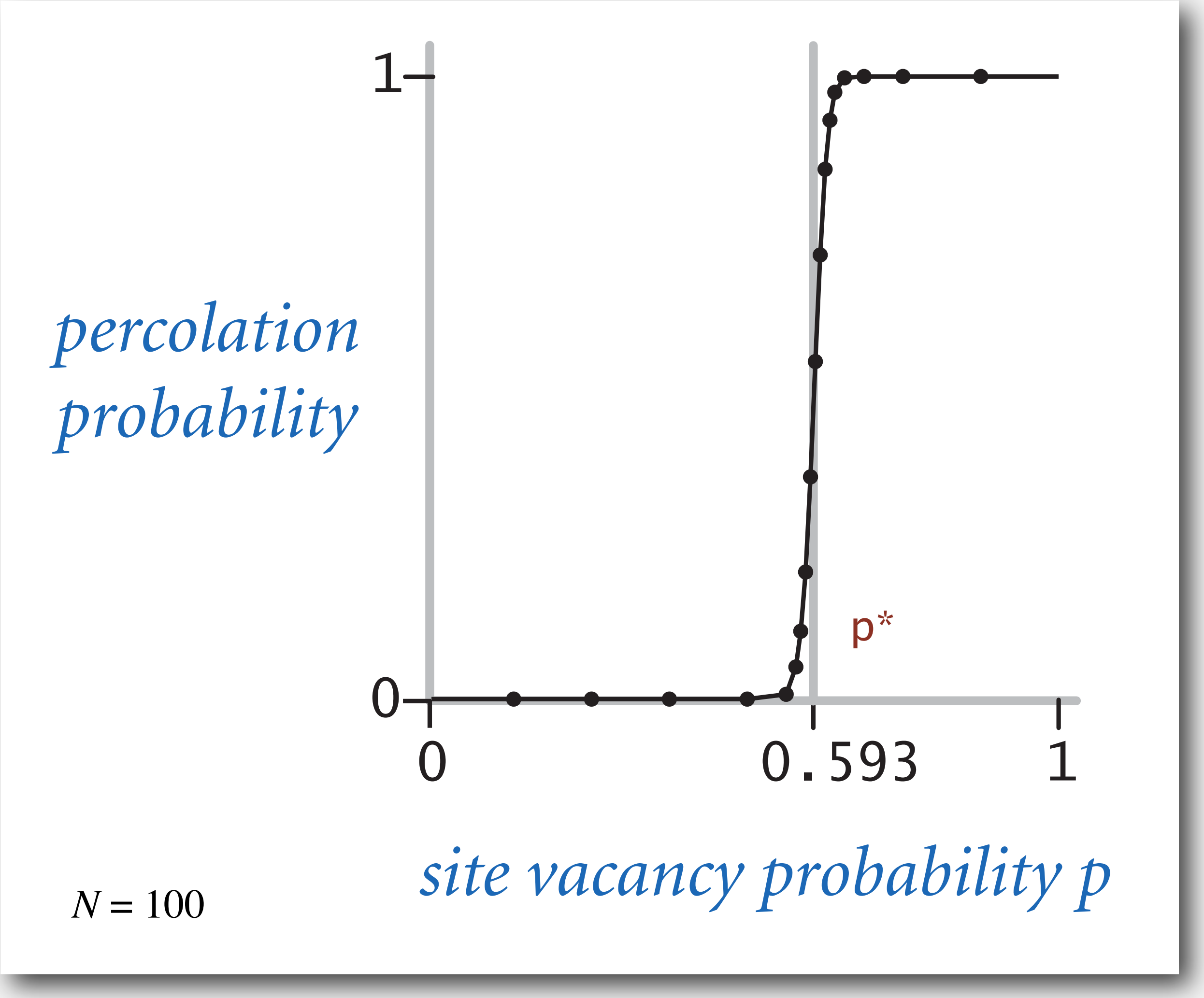
open site

blocked site

Percolation threshold

- Q. What is percolation threshold p^* ?
- A. About 0.592746 for large square lattices.

constant known only via simulation



- Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

- Steps to developing a usable algorithm.
  - Model the problem.
  - Find an algorithm to solve it.
  - Fast enough? Fits in memory?
  - If not, figure out why.
  - Find a way to address the problem.
  - Iterate until satisfied.

- The scientific method.

- Mathematical analysis.