Combinatorics

Coun@ng Principles

Pigeonhole Principle: \(k\) pairwise disjoint subsets of a set of \(n\) elements ...
\(\rightarrow\) at least one of them will have cardinality \(\geq \left\lceil \frac{n}{k} \right\rceil\)

Multiplication Principle: \(|A \times B \times C| = |A| \cdot |B| \cdot |C|

Addition Principle: \(|A \cup B \cup C| = |A| + |B| + |C|\)
(for pairwise disjoint sets only)

Inclusion/Exclusion Principle: \(|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|\)

Permutations and Combinations – Review from Chapter 6

\(P(n,k) = \text{k-permutations of a set with n elements}\)
- order matters
- Formula: \(P(n,k) = \frac{n!}{(n-k)!}\)

\(C(n,k) = \text{k-combinations of a set with n elements}\)
- order does NOT matter
- Formula: \(C(n,k) = \frac{n!}{(n-k)!k!}\)

Product rule
divide by \(k!\) due to overcounting

Pascal’s Triangle

\[\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
\end{array}\]
Pascal’s Triangle

- Pascal’s Triangle represents the identity:

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

for \(0 \leq k \leq n\).
Binomial Refresher

- A binomial expression is simply the sum of two terms
  - For example:
    - \((x+y)\)
    - \((x+y)^2\)
  - When a binomial expression is expanded, the binomial coefficients can be “seen”
    - For example:
      \[
      (x+y)^2 = x^2 + 2xy + y^2
      \]

Binomial Coefficients & Combinations

- Explore the following:
  \[
  (x+y)^3 = (x+y)(x+y)(x+y)
  \]
  
  \[
  xxx + xxy + xyy + yxx + yxy + yyx + yyy
  \]

  \[
  x^3 + 3x^2y + 3xy^2 + y^3
  \]

- Binomial Theorem
  \[
  (x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k
  \]

Some Corollaries of the Binomial Theorem

Corollary 1 \((a = b = 1)\):

Corollary 2 \((a = 1, b = -1)\):

Corollary 3 \((a = 1, b = 2)\):
Combinations with Repetition

- **Example:** How many ways are there to select any 4 fruits from a bowl containing oranges, apples, and bananas?

![Fruit combinations](http://www.pexels.com/photos-of-fruit-in-a-bowl/)

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Combinations with Repetition

- **Example:** How many ways are there to select 12 bills from a cash box containing $1, $5, $10, $20 and $50 bills?

![Stars & Bars Technique](http://www.pixell.club/pictures-of-fruit-in-a-bowl/)

<table>
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Combinations with Repetition

- **Theorem:**
  - There are \( C(k + n - 1, k) \) \( k \)-combinations from a set with \( n \) unique elements when repetition of elements is allowed.

- **Previous Example:**
  - \( n = 3 \) different fruits
  - \( k = 4 \) items to select
  - \( C(4+3-1, 4) = C(6,4) = \frac{6!}{(6-4)!4!} = 15 \)
### Combination with Repetition

**Example**
- How many ways can 6 balls be distributed into 9 different bins?

**Solution**
- \( n = 9 \) unique bins \( \Rightarrow 8 \) bars
- \( r = 6 \) balls \( \Rightarrow 6 \) stars
- 6+8 positions, choose 6 for the stars:
  - \( C(6+8,6) = C(14,6) \)

### Stars and Bars: Integer solutions

**Example:** \( a + b + c = 5 \)

- How many integer solutions with \( a,b,c \geq 0 \)?

- How many integer solutions with \( a,b,c \geq 1 \)?

### Example: Anagrams

*How many different strings can be made by reordering the letters of the string SUCCESS?*

**Solution:**
- 3 S’s, 2 C’s, 1 E and 1 U
- \( C(7,3) \) to place 3 S’s
- \( C(4,2) \) to place the C’s
- \( C(2,1) \) to place the E
- \( C(1,1) \) to place the U
- \( C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \times \frac{2!}{2!} \times \frac{1!}{1!} \times \frac{1!}{1!} \)

### k-Permutations with Repetition

The number of different permutations of \( k \) objects of \( n \) types, where there are \( k_1 \) identical objects of type 1, \( k_2 \) identical objects of type 2 ... and \( k_n \) identical objects of type \( n \), with \( k_1 + k_2 + k_3 + \ldots + k_n = k \):

\[
\frac{k!}{k_1!(k-k_1)!} \cdot \frac{(k-k_1)!}{k_2!(k-k_1-k_2)!} \cdot \ldots \cdot \frac{(k-k_1-k_2-\ldots-k_{n-1})!}{k_n!(k-k_1-k_2-\ldots-k_{n-1}-k_n)!}
\]

\[
= \frac{k!}{k_1!k_2!\ldots k_n!}
\]