Counting Principles

A faculty committee has decided to choose one or more students to join the committee. A total of 5 juniors and 6 seniors have volunteered to serve on this committee. How many different choices are there if the committee decides to select
(a) one junior and one senior?
(b) exactly one student?

Counting – Warmup Exercises

Multiplication Principle

- Cardinality of cartesian product of sets
- Choose an element from each of several sets

\[ |A \times B \times C| = |A| \cdot |B| \cdot |C| \]

Addition Principle

- Cardinality of union of disjoint sets
- Choose an element from one of several sets

\[ |A \cup B \cup C| = |A| + |B| + |C| \]
**Multiplication Principle - Example**

*Choose an element from each set – how many ways?*

Let $A = \{a, b, c\}$, $B = \{1, 2\}$

The *cartesian product* is the set of ordered pairs $(x, y)$ where $x \in A$ and $y \in B$:

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

**Multiplication Principle:** $|A \times B| = |A| \cdot |B|$

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**Addition Principle - Example**

*Choose an element from one of the disjoint sets – how many ways?*

Let $A = \{a, b, c\}$, $B = \{1, 2\}$

*Cardinality* of a set = number of members

- $|A| = 3$
- $|B| = 2$

$A \cup B = \{a, b, c, 1, 2\}$ \quad $A \cap B = \emptyset$

(A and B are *disjoint*)

**Addition Principle:** $|A \cup B| = |A| + |B|$

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**When to Use the Multiplication Principle**

- If an activity can be constructed of successive steps, to determine the possibilities, - Multiply together the number of ways of doing each step

- In other words, if
  - Step 1 = $n_1$ ways
  - Step 2 = $n_2$ ways
  - ....
  - Step $k = n_k$ ways

Then the number of possibilities = $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$

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**Multiplication Principle - Example**

- **Problem**
  - How many different license plates are available if each plate contains a sequence of 3 letters followed by three digits?

- **Solution**
  - 26 choices for each letter and 10 choices for each number

  $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$

  possible license plates
When to Use the Addition Principle

- If an activity can be constructed using several alternative ways, to determine the possibilities,
  - Sum together the choices for each of the alternative ways
- In other words, if
  - way 1 = $n_1$ choices
  - way 2 = $n_2$ choices
  - ... 
  - way k = $n_k$ choices
  Then the number of possibilities = $n_1 + n_2 + n_3 + ... + n_k$

Addition Principle – Example

- **Problem**
  - Your broker has told you to select a stock from one of the following lists:
    - 25 high tech companies
    - 15 consumer product companies
    - 10 service companies
  - How many choices do you have?
- **Solution**
  - $25 + 15 + 10 = 50$ choices
  *Note:* Must have disjoint sets of objects

Using both Principles

- **Problem**
  - Your broker has told you to select 2 stocks, each from a different list. You are given the following lists:
    - 25 high tech companies
    - 15 consumer product companies
    - 10 service companies
  - How many choices do you have?
- **Solution**
  - Decide which lists:
    1. high tech and service companies
    2. high tech list and consumer products
    3. consumer product and service companies
  - These are mutually exclusive choices, so you have $25*15 + 25*10 + 15*10 = 775$ choices
Using both Principles

- Problem
  How many bit strings of length 5 begin with 00 or with 11?
- Solution
  There are \(2^3\) five bit strings that begin with 00
  There are \(2^3\) five bit strings that begin with 11

Therefore, there are a total of \(2^3 + 2^3\) eight bit strings that begin with 00 or with 11.

Ice Cream Break

There are sugar cones, cake cones, and waffle cones;
There are five flavors of ice cream (vanilla, chocolate, strawberry, banana, pistachio).

How many different orders are there for cone/ice cream?
How many different orders for triple-decker cones?

- repeated flavors allowed
- no repeated flavors

Permutations

- A permutation of a non-empty set is an arrangement or ordered list of its elements
- There are \(n!\) permutations of a set of \(n\) elements
  - Easily proved using the Multiplication rule
- For example, 3 blocks can be ordered \(3! = 6\) ways
- 3 ice-cream flavors?
- 5 ice-cream flavors?

Permutations – Tree diagram
Permutations – Tree diagram

abc
acb
bac
bca
cab
cba

More Ice Cream
- 3 cones: sugar cones, cake cones, and waffle cones
- 5 flavors: vanilla, chocolate, strawberry, banana, pistachio

What if we add restrictions:
- must have pistachio or chocolate on top
- cannot have the banana at the bottom
- repeated flavors allowed
- no repeated flavors

Principle of Inclusion-Exclusion

The addition principle does not work if the sets are not disjoint (it counts the elements in their intersection twice).

- **Principle of Inclusion-Exclusion**: subtract the overcount

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

Compare with sum rule (when \( |A_1 \cap A_2| = 0 \)):

\[ |A_1 \cup A_2| = |A_1| + |A_2| \]

Problem: How many bit strings of length 5 begin with 0 or end with 11?

- 2^5 five bit strings that begin with 0
- 2^5 five bit strings that end with 11
- 2^5 five bit strings that begin with 0 **and** end with 11 (overcount)
Principle of Inclusion-Exclusion

- **Problem**: How many bit strings of length 5 begin with 0 or end with 11?

\[
\begin{align*}
\text{2}^4 & \text{ five bit strings that begin with 0} \\
\text{2}^4 & \text{ five bit strings that end with 11} \\
\text{2}^4 & \text{ five bit strings that begin with 0 and end with 11 (overcount)}
\end{align*}
\]

\[
\text{Total} = 2^4 + 2^4 - 2^2 \text{ five-bit strings that begin with 0 and end with 11.}
\]

Exercise

How many bit strings of length eight either begin with 111 or end with 00?

Principle of Inclusion-Exclusion for 3 sets?

\[|A_1 \cup A_2 \cup A_3| = \]

(A) How many students plan to take all three courses?
(B) How many students plan to take exactly one of the courses?
(C) How many students plan to take exactly two of the courses?
Principle of Inclusion-Exclusion for 4 sets?

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = \]

Pigeonhole principle

- \( n \) pigeons: 1 2 3 \ldots ? \ldots \( n \)
- \( k \) pigeonholes: 1 2 3 \ldots \( k \)

- if \( n > k \), then at least one pigeonhole has more than one pigeon
- if \( n > km \), then at least one pigeonhole has more than \( m \) pigeons
Pigeonhole principle

The Pigeonhole Principle: If a set $S$ with $n$ elements is divided into $k$ pairwise disjoint subsets $S_1, S_2, \ldots, S_k$, then at least one of these subsets has at least $\lceil n/k \rceil$ elements.

- How many students do you need to have in a class to ensure that you have two born on the same
  - month?
  - week?
  - same birthday?
- How about if you want to have at least 10 born on the same month?

Pigeonhole Principle - Example

- Given any list of 25 numbers, each of which has at most five digits, two subsets of the list have the same sum.

Pigeonhole Principle - Example

- Did you know that in San Francisco, at least four people have the same number of hairs on their heads?
  - Average human head: 100,000 - 180,000 hairs
  - San Francisco population > 700,000

Consider a standard deck of cards with suits hearts (♥), spades (♠), clubs (♣), and diamonds (♦), and values 2–10, jack, queen, king, and ace.

How many cards must you deal out before being assured that two will have the same suit?

How many must you deal out before being assured that two will have the same value?
Pigeonhole Principle - Example

• How many books must be chosen from 24 mathematics books, 25 computer science books, 21 literature books and 10 economics books in order to be certain there are at least 12 books on the same subject?

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Each piece of fruit in a fruit basket is either an apple, a banana, an orange, a pear or a peach. How many pieces of fruit must be in the basket to guarantee that there is at least one apple, at least two bananas, at least three oranges, at least four pears or at least five peach?

1 + (1 − 1) + (2 − 1) + (3 − 1) + (4 − 1) + (5 − 1) = 11.

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Permutations

• A permutation is an ordering of objects
  – For example, 3 blocks can be ordered 6 ways

• There are \( n! \) permutations of \( n \) elements
  – Easily proved using the Multiplication principle
Combinations

What if all that matters is which blocks you select, not the order?...

• A combination is an unordered selection of elements in a set

  Example: a 3-combination from a set of 12 colored blocks is simply a subset of cardinality 3.

\[ C(n, k) = \text{# of } k\text{-combinations of a set with } n \text{ elements} \]

Choice notation: \( \binom{n}{k} \)

Formula: \[ C(n, k) = \binom{n}{k} = \frac{n!}{(n - k)! \cdot k!} \]

k-Permutations

\[ P(n, k) = \text{# of } k\text{-permutations of a set with } n \text{ elements} \]

• The number of ways to permute \( k \) out of \( n \) items
• Similar to combinations, but order matters

• Formula:

\[ P(n, k) = \frac{n!}{(n - k)!} \]

Combinations

• Example: the number of ways to form a committee of 4 members from a department of 13 faculty
• denoted \( C(13, 4) \) or \( \binom{13}{4} \)

Permutations

• Example: the number of ways to choose 4 of the 13 faculty to teach upper level electives.

• denoted \( P(13, 4) \)
Exercises

Compute (a) $\binom{100}{20}$, (b) $P(8, 2)$, (c) $\binom{n}{2}$, (d) $P(A \cup B)$, (e) $P(n, n-1)$.

Watch out for wording:
How many different ways are there of selecting 5 people from a group of 100 people to serve on a panel?

vs.
How many different ways are there of selecting 5 people from a group of 100 people to serve on a panel and seating them in a row of 5 chairs?

Think about what you are counting:
How many subsets of {1, 2, 3, 4, 5, 6, 7, 8, 9}
• contain exactly three elements?
• contain exactly three elements, all of which are odd numbers?
• LIST THEM ALL!
There are 4 mathematics books, 3 computer science books and 2 engineering books to be placed on a book shelf.
• In how many ways can this be done?
• In how many ways can these books be placed on a book shelf if the books on the same subject must be grouped together

Properties of Combinations

\[ C(n, 0) = \frac{n!}{0!} \binom{n}{0} = 1 \text{ for any } n \geq 0 \]

\[ C(n, n) = \frac{n!}{n!} = 1 \]

\[ C(n, k) = C(n, n-k) \text{ for any } 0 \leq k \leq n \]

\[ C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n \text{ for any } n \geq 0 \]