Functions

CSC 1300 – Discrete Structures
Villanova University

Functions: Basic terminology

A function \( f \) from \( A \) to \( B \) assigns exactly one element of \( B \) to each element of \( A \).

\[ f : A \rightarrow B \]

We write \( f(x) = y \) if the function \( f \) assigns \( y \) to \( x \).

- The range of \( f \) is the set of all images of elements of \( A \).
- \( y \) is called the image of \( x \) (under \( f \)).
- The image of a subset \( S \) of \( A \), denoted by \( f(S) \), is the subset of \( B \) that consists of the images of the elements of \( S \):
  \[ f(S) = \{ f(x) \mid x \in S \} \]

Example: Let \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) be defined as \( f(x) = x + 1 \). Let \( E \) be the set of even integers. What is the image of \( E \)?

Functions: examples

- \( x^2 \), \( e^x \), |\( x \)|, \( \log x \), \( \ln x \)
- Floor \( \lfloor x \rfloor \)
- \( n! \)
- \( n \mod 5 \)
- \( |S| \) (where \( S \) is a finite subset of \( \mathbb{Z} \))
- ASCII table
- Identity function

One-to-One Functions

- A function \( f : X \rightarrow Y \) is one-to-one (or injective) iff for each \( y \in Y \) there is at most one \( x \in X \) with \( f(x) = y \).
- Examples:
  \[ \{(1,5), (2,3), (4,5)\} \quad f(x) = x^2 \text{ for } x \in \mathbb{Z} \]
Onto Functions

A function \( f : X \to Y \) is onto (or surjective) if for each \( y \in Y \) there exists an \( x \in X \) with \( f(x) = y \) (co-domain = range)

\[
\begin{array}{c|c|c}
\text{Onto} & \text{Neither one-to-one nor onto} & \text{One-to-one and onto} \\
\{(1,2), (2,4), (3,6), (4,6) \} & f(x) = x^2 \text{ for } x \in (1,0,1) \text{ and } y \in (1,0,2) & f(x) = x^2 \text{ for } x \in (1,0,2) \text{ and } y \in (1,0,4)
\end{array}
\]

Onto (but not one-to-one)

Neither one-to-one nor onto

One-to-one and onto (Bijection)

Inverse Function

- If a function \( f : A \to B \) is a bijection, the inverse function \( f^{-1} : B \to A \) is defined and is also a bijection mapping every \( y \in B \) to a unique \( x \in A \). Hence, \( f^{-1}(y) = x \) when \( f(x) = y \).

- Examples:
  - \( f = \{ (a,1), (b,2), (c,3) \} \)
  - \( f(x) = x+1, \ x \in \mathbb{Z} \)
  - \( f(x) = x^2, \ x \in \mathbb{R}^+ \)

Bijections

- A function \( f \) from \( X \) to \( Y \) is a bijection (or a one-to-one correspondence) if \( f \) is both one-to-one and onto (i.e., both injective and surjective).

Example: Let \( f(x) = x+1 \). Is \( f \) a bijection?
  - if the domain and codomain are \( \mathbb{N} \)
  - if the domain and codomain are \( \mathbb{Z} \)
  - if the domain and codomain are \( \mathbb{R} \)

Composition of Functions

The composition of the functions \( f : A \to B \) and \( g : B \to C \), denoted by \( g \circ f \), is defined by:

\[
(g \circ f)(x) = g(f(x))
\]

Note: the range of \( f \) must be a subset of the domain of \( g \).

Example: Let \( A = B = C = \mathbb{R}^+ \)
  - \( f(x) = 3x + 2 \)
  - \( g(x) = \frac{1}{x} \)

- If \( f \) and \( g \) are one-to-one, so is \( g \circ f \)
- If \( f \) and \( g \) are onto, so is \( g \circ f \)
- If \( f \) and \( g \) are bijections, so is \( g \circ f \)
- If \( f : A \to B \) is bijective, then
  - \( (f \circ f^{-1})(x) = x \) (identity on \( B \))
  - \( (f^{-1} \circ f)(x) = x \) (identity on \( A \))