Functions

CSC 1300 – Discrete Structures
Villanova University
Functions: Basic terminology

A function \( f \) from A to B assigns exactly one element of B to each element of A.

\[ f : A \rightarrow B \]

We write \( f(x) = y \) if the function \( f \) assigns \( y \) to \( x \)

- The range of \( f \) is the set of all images of elements of A.
- \( y \) is called the image of \( x \) (under \( f \)).
- The image of a subset \( S \) of A, denoted by \( f(S) \), is the subset of B that consists of the images of the elements of \( S \):
  \[ f(S) = \{ f(x) \mid x \in S \} \]

Example: Let \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) be defined as \( f(x) = x+1 \). Let \( E \) be the set of even integers. What is the image of \( E \)?
Functions: examples

- $x^2, e^x, |x|, \log_2 x, \ln x$
- Floor $\lfloor x \rfloor$
- $n!$
- $n \mod 5$
- $|S|$ (where $S$ is a finite subset of $\mathbb{Z}$)
- ASCII table
- Identity function
One-to-One Functions

• A function $f : X \rightarrow Y$ is **one-to-one** (or **injective**) iff for each $y \in Y$ there is at most one $x \in X$ with $f(x) = y$

• Examples:

  $\{(1,5), (2,3), (4,5) \}$  \hspace{1cm} $f(x) = x^2$ for $x \in \mathbb{Z}$

\[\begin{array}{c|c|c|c|c|c|c|c}
  & 1 & 2 & 4 & 5 & 6 \\
\hline
1 & 5 \\
2 & 3 \\
4 & 6
\end{array} \quad \begin{array}{c|c|c|c}
1 & 1 \\
0 & 0 \\
-1 & -1
\end{array}\]
Onto Functions

A function \( f : X \rightarrow Y \) is \textbf{onto} (or \textbf{surjective}) if for each \( y \in Y \) there exists an \( x \in X \) with \( f(x) = y \)
(co-domain = range)

\[ \{(1,2), (2,4), (3,6), (4,6) \} \]

\( f(x) = x^2 \) for \( x \in \{1,0,-1\} \)
and \( y \in \{1,0,2\} \)

\( f(x) = x^2 \) for \( x \in \{1,0,2\} \)
and \( y \in \{1,0,4\} \)

\textbf{Onto} (but not one-to-one)

\textbf{Neither one-to-one nor onto}

\textbf{One-to-one and onto (Bijection)}
Bijections

• A function $f$ from $X$ to $Y$ is a bijection (or a one-to-one correspondence) if $f$ is both one-to-one and onto (i.e., both injective and surjective).

Example. Let $f(x) = x + 1$. Is $f$ a bijection?

• if the domain and codomain are $\mathbb{N}$?
• if the domain and codomain are $\mathbb{Z}$?
• if the domain and codomain are $\mathbb{R}$?
Inverse Function

• If a function \( f: A \rightarrow B \) is a bijection, the inverse function \( f^{-1}: B \rightarrow A \) is defined and is also a bijection mapping every \( y \in B \) to a unique \( x \in A \). Hence, \( f^{-1}(y) = x \) when \( f(x) = y \).

• Examples:
  - \( f = \{ (a,1), (b,2), (c,3) \} \)
  - \( f(x) = x+1, \; x \in \mathbb{Z} \)
  - \( f(x) = x^2, \; x \in \mathbb{R}^+ \)
Composition of Functions

The composition of the functions \( f:A \to B \) and \( g:B \to C \), denoted by \( g \circ f \) is defined by:

\[
(g \circ f)(x) = g(f(x))
\]

Note: the range of \( f \) must be a subset of the domain of \( g \).

**Example:** Let \( A=B=C=\mathbb{R}^+ \)

\[
\begin{align*}
  f(x) &= 3x + 2 \\
  g(x) &= \frac{1}{x}
\end{align*}
\]

- If \( f \) and \( g \) are one-to-one, so is \( g \circ f \)
- If \( f \) and \( g \) are onto, so is \( g \circ f \)
- If \( f \) and \( g \) are bijections, so is \( g \circ f \)
- If \( f:A \to B \) is bijective, then
  - \( (f \circ f^{-1})(x) = f(f^{-1}(x)) = x \) (identity on \( B \))
  - \( (f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \) (identity on \( A \))