Relations

CSC 1300 – Discrete Structures
Villanova University
Relations: Basic terminology

Relations between elements of set $A$ and set $B$ can be expressed as sets of ordered pairs.

$$R : A \rightarrow B$$

domain co-domain

Example: The “hates” relation

A = \{ anne, bart, cal, dave \}
B = \{ arugula, broccoli, cauliflower, dill, escarole \}

$R = \{(anne, broccoli), (anne, cauliflower), (bart, arugula), (bart, broccoli), (bart, cauliflower), (bart, escarole), (dave, broccoli), (dave, cauliflower)\}$

So, bart $R$ arugula and cal $R$ arugula

- We write $x R y$ if $x$ is related to $y$ by the relation $R$
- The relation $R$ is a subset of $A \times B$
Relations: Basic terminology

A relation between elements of a set \( A \) is referred to as a relation on \( A \)

\[ R : A \rightarrow A \]

Example: The “likes” relation
\[ A = \{ \text{anne, bart, cal, dave} \} \]

\[ R = \{(\text{anne, bart}), (\text{anne, cal}), (\text{bart, cal}), (\text{cal, bart}), (\text{cal, cal}), (\text{dave, anne})\} \]

- We write \( x R y \) if \( x \) is related to \( y \) by the relation \( R \)
- The relation \( R \) is a subset of \( A \times A \)
Example: Relation on a set $A$

Relation on set $A = \{1,2,3,4\}$

$(x,y) \in R$ if $y \div x$ has remainder 0

(i.e., $x$ divides $y$)

$R = \{ (1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4) \}$
Properties of Relations

- **Reflexive**
  - for all \( a \in A \), \((a, a) \in R\)

- **Symmetric**
  - for all \( a, b \in A \), if \((a, b) \in R\), then \((b, a) \in R\)

- **Antisymmetric**
  - for all \( a, b \in A \), if \((a, b) \in R\) and \(a \neq b\), then \((b, a) \not\in R\)

- **Transitive**
  - for all \( a, b, c \in A \), if \((a, b) \in R\) and \((b, c) \in R\), then \((a, c) \in R\)

Given:
- the relation \( R \)
- the set \( A \)
Example: Relation on a set A

• **Example**
  – \((x,y) \in R\) if \(x\) divides \(y\) evenly on the set \(\{1,2,3,4\}\)

• **Solution**
  – \(R=\{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4)\}\)
Example: The “likes” relation

\[ A = \{ \text{anne, bart, cal, dave} \} \]

\[ R = \{ (\text{anne, bart}), (\text{anne, cal}), (\text{bart, cal}), (\text{cal, bart}), (\text{cal, cal}), (\text{dave, anne}) \} \]
Additional Properties

• A relation is called a **partial order** if it is reflexive, antisymmetric and transitive.
  – Partial orders represent relations that use ordered elements
  – Example: $\subseteq$ (subset relation)

• A relation is called an **equivalence relation** if it is reflexive, symmetric and transitive.
  – Formalizes the notion of “equivalence” or “sameness”.
  – Example: logical equivalence
Equivalence Relations

A relation on a set $A$ is called an *equivalence relation* if it is
(i) reflexive,
(ii) symmetric,
(iii) transitive.
Formalizes the notion of “equivalence” or “sameness”.

More Examples of equivalence relations:
- “to be equal” (for numbers or sets)
- “to have the same number of elements” (for sets)
- “to have the same age”
- “to have the same remainder after division by 2” (i.e., parity)
- “to have the same first 3 bits” (for bit strings)
- “to have the same truth table” (for propositions)
Equivalence classes

Let $R$ be an equivalence relation on a set $X$, and let $a$ be an element of $X$. The set of all elements of $X$ that are related to $a$ by $R$ is called the **equivalence class** for $a$ and is denoted by $[a]$. Any element $b \in [a]$, $b$ is called a **representative** of $[a]$.

**Example:** Let $R$ be a relation defined on $\mathbb{Z}$ by $a \mathrel{R} b$ if $a + b = 0$ or $a - b = 0$. Equivalence classes?

**Example:** Let $X$ be the set of all bit strings of length at least 3, and $R$ be the relation “agree in the first three bits”. Find
- $[001]$
- $[1101101]$
- $[0011]$
**Theorem.** Let $R$ be an equivalence relation on a set $X$. Then the equivalence classes of $R$ form a partition of $X$.

A *partition* of a set $X$ is a collection of disjoint nonempty subsets of $X$ that have $X$ as their union.

The collection $X_1, X_2, X_3, X_4$ is a partition of $X$. 
Partitioning?

**Example:** Let $R$ be a relation defined on $\mathbb{Z}$ by $a R b$ if $a + b = 0$ or $a - b = 0$.

**Example:** Let $X$ be the set of all bit strings of length at least 3, and $R$ be the relation “agree in the first three bits”.