1. Let $A = \{ x \mid x \in \mathbb{Z} \text{ and } -3 \leq x \leq 1 \}$ and $C = \{ x \in \mathbb{Z} \mid |x| < 4 \}$. Find all sets $B$ such that $A \subseteq B \subseteq C$.

**Answer:**
$A = \{-3, -2, -1, 0, 1\}$ and $C = \{-3, -2, -1, 0, 1, 2, 3\}$.

$B = \{-3, -2, -1, 0, 1\}$ or $B = \{-3, -2, -1, 0, 1, 3\}$

2. Give an example of three sets $A$, $B$, $C$ such that $A \subseteq C$, $B \subseteq C$, $A \nsubseteq B$, and $C \subseteq A$.

**Answer:** $A = \{ 1, 2 \}$; $B = \{ 1 \}$; $C = \{ 1, 2 \}$.

3. How many elements are in $\mathcal{P}(A)$ if $A = \{ x \in \mathbb{Z} \mid |x| < 6 \}$?

**Answer:** Since $|A| = 11$, it follows that $|\mathcal{P}(A)| = 2^{11} = 2048$.

4. For the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{I}$ (the integers, rational numbers, real numbers, and irrational numbers, respectively), where $\mathbb{R}$ is considered the universal set, determine the following:
   
   (a) $\mathbb{Z} \cap \mathbb{Q}$  
   (b) $\mathbb{I} \cap \mathbb{Q}$  
   (c) $\mathbb{Z} \cup \mathbb{Q}$  
   (d) $\mathbb{Z} - \mathbb{Q}$  
   (e) $\overline{\mathbb{Q}}$  
   (f) $\mathbb{I} \cup \mathbb{Q}$

**Answer:**
(a) $\mathbb{Z}$  
(b) $\emptyset$  
(c) $\mathbb{Q}$  
(d) $\emptyset$  
(e) $\overline{\mathbb{Q}}$  
(f) $\mathbb{R}$

5. For $n \in \mathbb{N}$, let $A_n = \left\{ \frac{n}{n+2} \right\}$ and $B_n = \left\{ \frac{1}{n} \right\}$. Determine

$$\left( \bigcup_{n=1}^{3} A_n \right) \cap \left( \bigcup_{n=1}^{3} B_n \right)$$

**Answer:** $\left\{ \frac{1}{3}, \frac{1}{2}, \frac{3}{5} \right\} \cap \left\{ 1, \frac{1}{2}, \frac{1}{3} \right\} = \left\{ \frac{1}{2}, \frac{1}{3} \right\}$
6. Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$.

(a) $(A \cup B) \times (A \cap B)$

*Answer:* $(A \cup B) \times (A \cap B) = \{(1,2), (2,2), (3,2), (4,2)\}$

(b) $(A \times B) \cap (B \times A)$

*Answer:* $(A \times B) \cap (B \times A) = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\} \cap \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3)\} = \{(2,2)\}$

(c) $(B - A) \times A$

*Answer:* $(B - A) \times A = \{4\} \times \{1,2,3\} = \{(4,1), (4,2), (4,3)\}$

(d) $(A \times A) \cap (B \times B)$

*Answer:* $(A \times A) \cap (B \times B) = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\} \cap \{(2,2), (2,4), (4,2), (4,4)\} = \{(2,2)\}$

7. Consider the universal set of all Villanova CS majors and define the following:

- $G$: the set of students that will graduate from Villanova after this semester
- $M$: the set of students that have fulfilled major and core requirements
- $P$: the set of students that have completed at least 40 courses of 3 credits or more

Express each of the sentences below in terms of sets, using appropriate set operations and set relations; draw Venn diagrams that exhibit the relations among the sets.

A. In order to graduate from Villanova at the end of this semester, you need to satisfy your major and core requirements, and to have completed at least 40 courses of 3 credits or more.

*Answer:* $G \subseteq M \cap P$
B. If you have fulfilled the major and core requirements and have completed 40 courses of 3 credits or more, then you will graduate from Villanova at the end of this semester.

**Answer:** \(( M \cap P ) \subseteq G \)

C. You will graduate from Villanova at the end of this semester if and only if you fulfill all the requirements and have completed at least 40 courses of 3 credits each.

**Answer:** \( M \cap P = G \)

D. If you have not completed 40 courses of 3 credits or more, you will not graduate from Villanova after this semester.

**Answer:** \( \overline{P} \subseteq \overline{G} \)
E. If you have completed 40 courses of 3 credits or more, then you have fulfilled all the major and core requirements
\[ \text{Answer: } P \subseteq M \]

F. If you have fulfilled all the major and core requirements, then you will graduate from Villanova at the end of this semester
\[ \text{Answer: } M \subseteq G \]

G. If you are a CS major and have not completed 40 courses of 3 credits or more, then you can’t have fulfilled all the major and core requirements
\[ \text{Answer: } \overline{P} \subseteq \overline{M} \]

H. Fulfilling all the major and core requirements is sufficient for completing 40 courses of 3 credits or more.
\[ \text{Answer: } M \subseteq P \]
8. Prove that if $A \subseteq B$, then $A \cap B = A$.

**Answer:**
There are two parts. We show first that $A \subseteq B \implies A \cap B = A$.
Suppose $A \subseteq B$. Clearly $A \subseteq A \cap B$, so now we need to show that $A \cap B \subseteq A$ is also true if $A \subseteq B$. Since, by definition, any element of $A \cap B$ is in $A$ and $B$, and all elements of $A$ are also in $B$, it follows that every element of $A \cap B$ is an element of $A$, thus $A \cap B \subseteq A$. Since $A \subseteq A \cap B$ and $A \cap B \subseteq A$, we have that $A \cap B = A$.

Next we show that $A \cap B = A \implies A \subseteq B$.
Suppose $A \cap B = A$. Thus $A \cap B \subseteq A$, so any element in $A$ and in $B$, is also an element of $A$, from which it follows that $A \subseteq B$. 