Graph Planarity

CSC 1300 – Discrete Structures
Villanova University

Major Themes

• Planar and non-planar graphs
• The graphs $K_5$ and $K_{3,3}$
• Dividing the plane into regions
• Euler’s formula

Three house/three utility puzzle

• Is it possible to join three houses to three separate utilities so that none of the connections cross?

Planar graphs

A graph is planar if it can be drawn in the plane without any edges crossing
K₃,₃ is not planar

- Consider region R formed by vertices 1, 4, 2, 5.
- **Case 1:** vertex 3 inside R
- **Case 2:** vertex 3 outside R

Where can vertex 6 go?

K₅ is not planar

Prove using a similar argument based on regions.
Kuratowski’s Theorem
Are there non-planar graphs other than $K_{3,3}$ and $K_5$?

What makes a graph non-planar?
- If a graph has $K_{3,3}$ or $K_5$ as a subgraph
- If a graph has a subgraph that can be obtained by adding “intermediate” vertices to edges of either $K_{3,3}$ or $K_5$ (such a graph is called a subdivision)
- any others?

Kuratowski’s Theorem:
A graph is nonplanar if and only if it contains a subgraph that is a subdivision of either $K_{3,3}$ or $K_5$.

Regions of the plane
- Planar graphs divide the plane into regions
Consider the following problem involving rice fields separated by walls. How many edges (walls) can we remove before all the rice fields are flooded? What type of graph are we left with after all these walls are removed?

Example: Relating number of regions to vertices and edges

Number of regions: \( r = \) 
Number of vertices: \( n = \) 
Number of edges: \( e = \) 
Edges removed in flood: \( e_1 = \) 
Edges remaining: \( e_2 = \)

Euler’s Identity for planar graphs

**Theorem:** Let \( G \) be a connected planar graph with \( n \) vertices, \( e \) edges and \( r \) regions. Then: 
\[ n - e + r = 2 \]

A corollary of Euler’s Identity helps rule out planarity for some graphs

**Corollary:** If \( G \) is connected planar graph, then 
\[ e \leq 3n - 6 \]

Intuition: graphs with lots of edges cannot be planar!

Example: What can we say about \( K_5 \) and \( K_{3,3} \)?
Previous examples

\[ e = \]
\[ n = \]
\[ e > 3n - 6 \, ? \]

Planar?

\[ e = \]
\[ n = \]
\[ e > 3n - 6 \, ? \]

Planar?