Graphs

Graphs are discrete structures consisting of vertices and edges that connect these vertices.

Graphs can be used to model:
- computer systems/networks
- mathematical relations
- logic circuit layout
- jobs/processes

Questions
- isomorphism
- cycles/paths
- planarity
- coloring

Graphs: Basic Terminology

- A **graph** is defined as $G = (V, E)$ with the set of vertices $V$ and a set of edges $E$.
- Two vertices $u$ and $v$ in an undirected graph $G$ are **adjacent** (or **neighbors**) if $\{u, v\}$ is an edge of $G$.
  - The edge $e$ is said to **connect** (or to be **incident**) with $u$ and $v$.
  - **Order** of a graph = number of vertices

Directed Graphs

- By definition, the edges of a directed graph are ordered pairs.
- In a directed graph, if we have edge $e = (u, v)$, then
  - $u$ is said to be adjacent to $v$, the **terminal vertex**
  - $v$ is said to be adjacent from $u$, the **initial vertex**
# Types of Graphs

<table>
<thead>
<tr>
<th>Type</th>
<th>Edges</th>
<th>Multiple Edges?</th>
<th>Loops?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple graph</td>
<td>Undirected</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Multigraph</td>
<td>Undirected</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Pseudograph</td>
<td>Undirected</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Directed graph</td>
<td>Directed</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Directed multigraph</td>
<td>Directed</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Terminology varies – be sure to check definitions when consulting other books/articles.

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# Degree of a vertex

- The **degree of a vertex** $v$ is the number of edges incident on $v$.
  - Denoted $\deg(v)$
  - A loop on $v$ contributes 2 to the $\deg(v)$

- Example:

  ![Example Graph](image)

  - $\deg(a) = 3$
  - $\deg(b) = 1$
  - $\deg(c) = 5$
  - $\deg(e) = 0$
  - $\deg(d) = 1$

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# Handshaking Lemma

In an undirected graph, the sum of the degrees of the vertices is twice the number of edges. Therefore, the sum of the degrees of all the vertices is even.

\[
\sum_{v \in V} \deg(v) = 2|E|.
\]

- **Corollary:** An undirected graph has an even number of vertices of odd degree.

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# Path

- A **walk** begins at vertex $v_0$, follows an edge $e_1$ to $v_1$, follows another edge to $v_2$ ...
  - A walk is represented without edges when there are no parallel edges
    - $(v_0, v_1, v_2, \ldots v_n)$
  - Said to be of **length** $n$
- A **trail** from $v_0$ to $v_n$ is a walk with no repeated edges.
- A **path** from $v_0$ to $v_n$ is a walk with no repeated vertices.
- A walk is **closed** if it starts and ends in the same vertex; otherwise the walk is said to be **open**
Circuits and cycles

- A **circuit** is a closed trail of length greater than 2.

  - A **cycle** (or **simple circuit**) is a circuit from \( v \) to \( v \) with no repeated vertex, except \( v \).
    - **3-cycle** or **triangle**: a cycle of length 3
    - **k-cycle**: cycle of length \( k \)

- Some cycles:
  - \( (b,c,d,e,b) \)
  - \( (b,a,c,d,e,a) \)
  - \( (b,c,a,d,e,b) \)
  - \( (c,d,e,a,b,c) \)

Connected Graph

- A graph \( G \) is **connected** if given any vertices \( v_1 \) and \( v_2 \) in \( G \), there is a path from \( v_1 \) to \( v_2 \).

Bipartite Graph

- A simple graph is called **bipartite** if its vertex set \( V \) can be partitioned into 2 disjoint sets \( V_1 \) and \( V_2 \) such that every edge in the graph connects a vertex in \( V_1 \) to a vertex in \( V_2 \).

  - **Bipartite**
    - \( V_1 = \{b,a,e\} \)
    - \( V_2 = \{c,d\} \)

  - **Not Bipartite**

Additional concepts (see text)

- **multipartite**
- **bridge**
- **cut-vertex**
- **disconnected**
- **connected component**
### Special Graphs
- Cycle \( C_n \)
- Cycle \( P_n \)
- Wheels \( W_n \)
- Complete graph \( K_n \)
- Complete Bipartite \( K_{n,m} \)
- 2-regular: all vertices have degree 2
- \( n \)-regular: all vertices have degree \( n \).
- Tree: a connected graph with no cycles
- Forest: a graph with no cycles

### Complement
Let \( G = (V, E) \)
- The complement \( \overline{G} = (V, E') \)
  - where \( E' = \{ \{u,v\} \mid u \in V \text{ and } v \in V \text{ and } \{u,v\} \notin E \} \)
- **Example:** What is the complement of \( G_1 \)?

### Union
The **union** of \( G_1=(V_1,E_1) \) and \( G_2=(V_2,E_2) \) is the graph:
\[
G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)
\]
**Example:**

### Subgraph
- \( H \) is a **subgraph** of \( G \) iff \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \)
- **Example:** \( G_2 \) is a subgraph of \( G_1 \)

A **connected component** of a graph is a subgraph that is connected.
Subgraphs

- H is an **induced subgraph** of G iff whenever two vertices of H are adjacent in G, they are also adjacent in H.
  - In other words, H consists of a subset of G’s vertices and all the edges between them.
- A subgraph H of a graph G is a **spanning subgraph** of G iff V(H) = V(G).

Isomorphism...

- It means two graphs are essentially the same (maybe drawn differently and re-labeled)
  - Same number of vertices
  - Same number of edges
  - Same degree sequence

Representation of Graphs

<table>
<thead>
<tr>
<th>Adjacency list</th>
<th>Adjacency matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>{{a, b, c, d, e}, {a,b}, {b,c}, {c,d}, {e,d}, {e,b}}</td>
<td></td>
</tr>
</tbody>
</table>

Isomorphism

- Definition: Two graphs $G_1$, $G_2$ are **isomorphic** iff there exists a bijection $h : V(G_1) \rightarrow V(G_2)$ such that
  - $\{v, w\} \in E(G_1)$ iff $\{h(v), h(w)\} \in E(G_2)$
  - In other words, $h$ is a bijection between the vertices of the two graphs that preserves edges.

Sometimes it is hard to tell... are these essentially the same graph?
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If we renumber the vertices, the graphs are still isomorphic, but we get a different isomorphism function.

Isomorphism...

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  - Same number of edges
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Sometimes it is hard to tell... are these essentially the same graph?

Isomorphism function:
- $h(1) = 4$
- $h(2) = 3$
- $h(3) = 6$
- $h(4) = 2$
- $h(5) = 1$
- $h(6) = 5$

Sometimes it is hard to tell... are these essentially the same graph?

NO – $G_1$ is 3-regular, whereas $G_2$ is not (e.g., vertex 5 has degree 2)