Counting Principles
Part 2

CSC 1300 – Discrete Structures
Villanova University
Counting Principles

**Multiplication Principle:** \(|A \times B \times C| = |A| \cdot |B| \cdot |C|\)

**Addition Principle:** \(|A \cup B \cup C| = |A| + |B| + |C|\)
(for pairwise disjoint sets only)

**Inclusion/Exclusion Principle:** \(|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|

**Pigeonhole Principle:** \(k\) pairwise disjoint subsets of a set of \(n\) elements ...

\(\Rightarrow\) at least one of them will have cardinality \(\geq \left\lfloor \frac{n}{k} \right\rfloor\)
Permutations and Combinations – Review from Chapter 6

\[ P(n,k) = \text{k-permutations of a set with n elements} \]

- order matters
- Formula: \( P(n,k) = \frac{n!}{(n-k)!} \)

\[ C(n,k) = \text{k-combinations of a set with n elements} \]

- order does NOT matter
- Formula: \( C(n,k) = \frac{n!}{(n-k)!k!} \)

divide by k! due to overcounting
Pascal’s Triangle

• Pascal’s Triangle represents the identity:

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

for \(0 \leq k \leq n\)
Binomial Refresher

• A binomial expression is simply the sum of two terms
  – For example:
    • (x+y)
    • (x+y)^2

• When a binomial expression is expanded, the binomial coefficients can be “seen”
  – For example:
    
    \[(x+y)^2 = x^2 + 2xy + y^2\]
    
    \[= 1x^2 + 2xy + 1y^2\]
Binomial Coefficients & Combinations

• Explore the following:

\[(x+y)^3 = (x+y)(x+y)(x+y)\]

\[xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\]

\[x^3 + 3x^2y + 3xy^2 + y^3\]

\[C(3,0) \quad C(3,1) \quad C(3,2) \quad C(3,3)\]

• Binomial Theorem

\[(x+y)^n = \sum_{k=0}^{n} C(n,k) x^{n-k} y^k\]
Binomial Theorem

• What is the expansion of \((x+y)^4\)?

• What is the term containing \(x^4\) in the expansion of \((x+y)^{10}\)?

• Find the coefficient \(x^4y^7\) in the expansion of \((2x+y)^{11}\).
Some Corollaries of the Binomial Theorem

Corollary 1 \((a = b = 1)\):

Corollary 2 \((a = 1, b = -1)\):

Corollary 3 \((a = 1, b = 2)\):
Combinations with Repetition

• **Example:** How many ways are there to select any 4 fruits from a bowl containing oranges, apples, and bananas?

http://www.pixell.club/pictures-of-fruit-in-a-bowl/
Combinations with Repetition

• **Example:** How many ways are there to select 12 bills from a cash box containing $1, $5, $10, $20 and $50 bills?

<table>
<thead>
<tr>
<th>$1</th>
<th>$5</th>
<th>$10</th>
<th>$20</th>
<th>$50</th>
</tr>
</thead>
</table>

*** | ***** | *** | *

1 2 3 4

Stars & Bars Technique
Stars & Bars Technique: try with the fruit: (fill in remaining combinations)

<table>
<thead>
<tr>
<th>Apple</th>
<th>Banana</th>
<th>Orange</th>
</tr>
</thead>
<tbody>
<tr>
<td>🍏🍏🍏🍏</td>
<td>🍌🍌🍌🍌</td>
<td>🍊🍊🍊🍊</td>
</tr>
<tr>
<td>🍏🍏🍏🍌</td>
<td>🍏</td>
<td>🍌🍌🍌</td>
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</tr>
</tbody>
</table>

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Combinations with Repetition

• Theorem:
  – There are $C(k + n - 1, k)$ $k$-combinations from a set with $n$ unique elements when repetition of elements is allowed.

• Previous Example:
  – $n = 3$ different fruits
  – $k = 4$ items to select
  – $C(4+3-1, 4) = C(6,4) = 6!/[((6-4)!4!)] = 15$
Combination with Repetition

• Example
  – How many ways can 6 balls be distributed into 9 different bins?

• Solution
  – $n = 9$ unique bins $\Rightarrow 8$ bars
  – $r = 6$ balls $\Rightarrow 6$ stars
  6+8 positions, choose 6 for the stars:
  – $C(6+8, 6) = C(14, 6)$
Stars and Bars: Integer solutions

Example:  \( a + b + c = 5 \)

• How many integer solutions with \( a, b, c \geq 0 \)?

• How many integer solutions with \( a, b, c \geq 1 \)?
Example: Anagrams

How many different strings can be made by reordering the letters of the string SUCCESS?

Solution:

– 3 S’s, 2 C’s, 1 E and 1 U
– C(7,3) to place to S’s
– C(4,2) to place the C’s
– C(2,1) to place the E
– C(1,1) to place the U
– C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \frac{4!}{2!2!} \frac{2!}{1!1!} \frac{1!}{1!0!}
**k-Permutations with Repetition**

The number of different permutations of \( k \) objects of \( n \) types, where there are \( k_1 \) identical objects of type 1, \( k_2 \) identical objects of type 2 ... and \( k_n \) identical objects of type \( n \), with \( k_1 + k_2 + k_3 + \ldots + k_n = k \):

\[
\binom{k}{k_1} \cdot \binom{k-k_1}{k_2} \cdot \binom{k-k_1-k_2}{k_3} \ldots \binom{k-k_1-k_2-\ldots-k_{n-1}}{k_n}
\]

\[
= \frac{k!}{k_1!(k-k_1)!} \cdot \frac{(k-k_1)!}{k_2!(k-k_1-k_2)!} \cdot \frac{(k-k_1-k_2)!}{k_3!(k-k_1-k_2-k_3)!} \ldots \frac{(k-k_1-k_2-\ldots-k_{n-1})!}{k_n!(k-k_1-k_2-\ldots-k_{n-1}-k_n)!}
\]

\[
= \frac{k!}{k_1! \cdot k_2! \cdot k_2! \cdots k_n!}
\]