Permutations and Combinations

CSC 1300 – Discrete Structures
Villanova University

Permutations

- A permutation is an ordering of objects
  - For example, 3 blocks can be ordered 6 ways

There are $n!$ permutations of $n$ elements
- Easily proved using the Multiplication principle

Combinations

- What if all that matters is which blocks you select, not the order?
- A combination is an unordered selection of elements in a set
- Example: a 3-combination from a set of 12 colored blocks is simply a subset of cardinality 3.

$C(n,k) = \# \text{ of } k\text{-combinations of a set with } n \text{ elements}$

Choice notation:

$n \choose k$

Formula:

$C(n, k) = \binom{n}{k} = \frac{n!}{(n - k)! \cdot k!}$

Example: the number of ways to form a committee of 4 members from a department of 13 faculty

- denoted $C(13,4)$ or $\binom{13}{4}$
**k-Permutations**

\[ P(n,k) = \text{# of } k\text{-permutations of a set with } n \text{ elements} \]

- The number of ways to permute \( k \) out of \( n \) items
- Similar to combinations, but \textbf{order matters}

- Formula:
  \[ P(n,k) = \frac{n!}{(n-k)!} \]

**Permutations**

- \textbf{Example:} the number of ways to choose 4 of the 13 faculty to teach upper level electives.
- denoted \( P(13,4) \)

**Exercises**

Compute (a) \( \frac{100!}{53!} \), (b) \( P(8,2) \), (c) \( \frac{6!}{2!} \), (d) \( \frac{P(7,3)}{P(7,4)} \), (e) \( \frac{m!}{(n-1)!} \).

**Exercises**

\textbf{Watch out for wording:}

How many different ways are there of selecting 5 people from a group of 100 people to serve on a panel?

\textbf{vs.}

How many different ways are there of selecting 5 people from a group of 100 people to serve on a panel and seating them in a row of 5 chairs?
**Exercises**

*Think about what you are counting:*
How many subsets of \{1, 2, 3, 4, 5, 6, 7, 8, 9\} contain exactly three elements?

- contain exactly three elements, all of which are odd numbers?

- list them all!

There are 4 mathematics books, 3 computer science books and 2 engineering books to be placed on a book shelf.

- In how many ways can this be done?

- In how many ways can these books be placed on a book shelf if the books on the same subject must be grouped together

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**Properties of Combinations**

\[ C(n,0) = \quad \text{for any } n \geq 0 \]

\[ C(n,n) = \]

\[ C(n,k) = C(n, n-k) \quad \text{for any } 0 \leq k \leq n \]

\[ C(n,0) + C(n,1) + \ldots + C(n,n) = \quad \text{for any } n \geq 0 \]

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**Five friends went on a Spring Break holiday together....**

They would like to have pictures with all combinations of them to post on social media. They already have a picture with all of them (at the airport) and none of them (view from the hotel).

- How many additional pictures do they need to ensure they have pictures of themselves in all possible groupings of:
  - 1 person
  - 2 people
  - 3 people
  - 4 people

- Now compute the total number of additional pictures in two ways:
  - (i) by summing the above numbers
  - (ii) by computing the total number of additional pictures based on the cardinality of the set of friends

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**Back to school...**

Villanova Spring Break is over and it is time to fly back. The seat assignment on the aircraft has the five friends occupying a row with three seats and another two across the aisle. Of course, they are free to switch around among these seats.

- How many ways can the friends arrange themselves into the assigned seats?

- How many ways can the friends be divided into the two groups (each side of aisle).

- How does that relate to the total seating arrangements?

- If we generalize the problem to \( n \) friends divided into two groups of \( k \) and \( n-k \) on either side of the aisle. Consider the identity:

\[ \binom{n}{k} \cdot k! \cdot (n-k)! = P(n,n) \]