Counting Principles

A faculty committee has decided to choose one or more students to join the committee. A total of 5 juniors and 6 seniors have volunteered to serve on this committee. How many different choices are there if the committee decides to select

(a) one junior and one senior?

(b) exactly one student?

Counting – Warmup Exercises

Multiplication Principle

- Cardinality of cartesian product of sets
- Choose an element from each of several sets

\[ |A \times B \times C| = |A| \cdot |B| \cdot |C| \]

Addition Principle

- Cardinality of disjoint union of sets
- Choose an element from one of several sets

\[ |A \cup B \cup C| = |A| + |B| + |C| \]
**Multiplication Principle - Example**

Let \( A = \{a, b, c\}, \quad B = \{1, 2\} \)
The *cartesian product* is the set of ordered pairs \((x, y)\) where \(x \in A\) and \(y \in B\):

\[
A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}
\]

**Multiplication Principle:** \(|A \times B| = |A| \cdot |B|

**Addition Principle - Example**

Let \( A = \{a, b, c\}, \quad B = \{1, 2\} \)

*Cardinality* of a set = number of members

\(|A| = 3\)
\(|B| = 2\)

\(A \cup B = \{a, b, c, 1, 2\} \quad A \cap B = \emptyset\)

(A and B are disjoint)

**Addition Principle:** \(|A \cup B| = |A| + |B|

**When to Use the Multiplication Principle**

- If an activity can be constructed of successive steps, to determine the possibilities,
  - Multiply together the number of ways of doing each step

- In other words, if
  
  Step 1 = \(n_1\) ways
  
  Step 2 = \(n_2\) ways
  
  ....
  
  Step \(k\) = \(n_k\) ways

  Then the number of possibilities = \(n_1 \cdot n_2 \cdot n_k \cdots n_k\)

**Multiplication Principle - Example**

- **Problem**
  - How many different license plates are available if each plate contains a sequence of 3 letters followed by three digits?

- **Solution**
  - 26 choices for each letter and 10 choices for each number

  \[
  \text{possible license plates} = 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000
  \]
When to Use the Addition Principle

- If an activity can be constructed of using several alternative ways, to determine the possibilities,
  - Sum together the choices for each of the alternative ways
- In other words, if
  - way 1 = \( n_1 \) choices
  - way 2 = \( n_2 \) choices
  - ... 
  - way \( k \) = \( n_k \) choices

Then the number of possibilities = \( n_1 + n_2 + n_3 + ... + n_k \)

Addition Principle – Example

- **Problem**
  - Your broker has told you to select a stock from one of the following lists:
    - 25 high tech companies
    - 15 consumer product companies
    - 10 service companies
  - How many choices do you have?

- **Solution**
  - \( 25 + 15 + 10 = 50 \) choices

*Note: Must have disjoint sets of objects*

Using both Principles

- **Problem**
  - Your broker has told you to select 2 stocks, each from a different list. You are given the following lists:
    - 25 high tech companies
    - 15 consumer product companies
    - 10 service companies
  - How many choices do you have?

- **Solution**
  - Decide which lists:
    1. high tech and service companies
    2. high tech list and consumer products
    3. consumer product and service companies
  - These are mutually exclusive choices, so you have \( 25 \times 10 + 25 \times 15 + 15 \times 10 = 775 \) choices

Using both Principles

- **Problem**
  - How many strings are there of lowercase alpha characters of length four or less?

- **Solution**
  - There are 26 lowercase alpha characters
    - For 4 characters, there are \( 26 \times 26 \times 26 \times 26 = 456,976 \) possible strings
    - For 3 characters, there are \( 26 \times 26 \times 26 = 17,576 \) possible strings
    - For 2 characters, there are \( 26 \times 26 = 676 \) possible strings
    - For 1 character, there are 26 possible strings
    - Thus the total \( 456,976 + 17,576 + 676 + 26 = 475,254 \)
  - Let’s not forget the empty string so we have a total of 475,255 possible strings
Ice Cream Break

A group of friends goes out for single-scoop ice-cream cones. There are sugar cones, cake cones, and waffle cones. There are five flavors of ice cream (vanilla, chocolate, strawberry, banana, pistachio).

How many different orders are there for cone/ice cream?

How many different orders for triple-decker cones?

• repeated flavors allowed
• no repeated flavors
• What if we add these restrictions:
  – must have pistachio or chocolate on top
  – cannot have the banana at the bottom

Using both Principles

• Problem
  How many bit strings of length 5 begin with 00 or with 11?

• Solution
  There are $2^5$ five bit strings that begin with 00
  There are $2^3$ five bit strings that begin with 11

Therefore, there are a total of $2^5 + 2^3 = 8$ eight bit strings that begin with 00 or with 11.

Principle of Inclusion-Exclusion

The addition principle does not work if the sets are not disjoint (it counts the elements in their intersection twice).

• Principle of Inclusion-Exclusion: subtract the overcount

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

Compare with sum rule (when $|A_1 \cap A_2| = 0$):

\[ |A_1 \cup A_2| = |A_1| + |A_2| \]
Principle of Inclusion-Exclusion

- **Problem**: How many bit strings of length 5 begin with 0 or end with 11?

Total = \(2^4 + 2^3 - 2^2\) five-bit strings that begin with 0 and end with 11.

Principle of Inclusion-Exclusion for 3 sets?

\[|A_1 \cup A_2 \cup A_3| = \]

Principle of Inclusion-Exclusion

- **Problem**: How many bit strings of length eight either begin with 111 or end with 00?

Principle of Inclusion-Exclusion

A total of 36 students plan to take at least one of the courses Discrete Mathematics, Algebra and Calculus during the coming semester. Of these 36 students, it is known that

- 23 students plan to take Discrete Mathematics,
- 19 students plan to take Algebra,
- 18 students plan to take Calculus,
- 7 students plan to take Discrete Mathematics and Algebra,
- 9 students plan to take Discrete Mathematics and Calculus and
- 11 students plan to take Algebra and Calculus.

(a) How many students plan to take all three courses?
(b) How many students plan to take exactly one of the courses?
(c) How many students plan to take exactly two of the courses?
Principle of Inclusion-Exclusion for 4 sets?

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = \]