Functions

CSC 1300 – Discrete Structures
Villanova University

Functions: Basic terminology

A function \( f \) from \( A \) to \( B \) assigns exactly one element of \( B \) to each element of \( A \).

\[ f : A \rightarrow B \]

We write \( f(a) = y \) if the function \( f \) assigns \( y \) to \( x \).

- The range of \( f \) is the set of all images of elements of \( A \).
- \( y \) is called the image of \( x \) (under \( f \)).
- The image of a subset \( S \) of \( A \), denoted by \( f(S) \), is the subset of \( B \) that consists of the images of the elements of \( S \):
  \[ f(S) = \{ f(x) \mid x \in S \} \]

Example: Let \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) be defined as \( f(x) = x + 1 \). Let \( E \) be the set of even integers. What is the image of \( E \)?

One-to-One Functions

- A function \( f : X \rightarrow Y \) is one-to-one (or injective) iff for each \( y \in Y \) there is at most one \( x \in X \) with \( f(x) = y \).
- Examples:
  \[ \{(1,5), (2,3), (4,5)\} \]
  \[ f(x) = x^2 \text{ for } x \in \mathbb{Z} \]
A function $f : X \to Y$ is onto (or surjective) if for each $y \in Y$ there exists an $x \in X$ with $f(x) = y$ (co-domain = range)

onto functions:

- $(1,2), (2,4), (3,6), (4,6)$
- $f(x) = x^2$ for $x \in (1,0,1)$ and $y \in (1,0,2)$
- $f(x) = x^2$ for $x \in (1,0,2)$ and $y \in (1,0,4)$

not one-to-one:

- $f(x) = x+1$, $x \in \mathbb{Z}$
- $f(x) = x^2$, $x \in \mathbb{R}^+$

Ex. Let $f(x) = x+1$. Is $f$ a bijection?
- if the domain and codomain are $\mathbb{N}$?
- if the domain and codomain are $\mathbb{Z}$?
- if the domain and codomain are $\mathbb{R}$?

One-to-one and onto (Bijection)

- $f(x) = x^2$ for $x \in (1,0,2)$ and $y \in (1,0,4)$

If a function $f : A \to B$ is a bijection, the inverse function $f^{-1} : B \to A$ is defined and is also a bijection mapping every $y \in B$ to a unique $x \in A$.

Hence, $f^{-1}(y) = x$ when $f(x) = y$.

- $f = \{(a,1), (b,2), (c,3)\}$
- $f(x) = x+1$, $x \in \mathbb{Z}$
- $f(x) = x^2$, $x \in \mathbb{R}^+$

composition of functions:

The composition of the functions $f : A \to B$ and $g : B \to C$, denoted by $g \circ f$, is defined by:

$(g \circ f)(x) = g(f(x))$

Note: the range of $f$ must be a subset of the domain of $g$.

Example: Let $A = \mathbb{N}$, $B = \mathbb{R}^+$, $f(x) = 3x + 2$, $g(x) = 1/x$

- if $f$ and $g$ are one-to-one, so is $g \circ f$
- if $f$ and $g$ are onto, so is $g \circ f$
- if $f$ and $g$ are bijections, so is $g \circ f$
- if $f : A \to B$ is bijective, then
  - $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ (identity on B)
  - $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ (identity on A)