Mathematical Induction

CSC 1300 – Discrete Structures
Villanova University
Mathematical Induction

• Use to prove universal statements of the form:
  – \( \forall n \ P(n) \)

• Easy to think of it as the domino effect
  Let \( P(n) \) be the statement: *domino n is knocked over*

  – If the first domino is knocked over (\( P(1) \) is true); AND
  – if whenever the \( k \text{th} \) domino is knocked over, then the \( (k + 1) \text{th} \) is also knocked over
    i.e., \( P(k) \Rightarrow P(k + 1) \) is true for every \( k \)

  …… *It follows that all the dominoes are knocked over.*
  (i.e., \( \forall n \ P(n) \) is true)
Principle of Mathematical Induction

\[ [ P(1) \land \forall k (P(k) \rightarrow P(k+1)) ] \rightarrow \forall n P(n) \]

Base Case
Inductive Hypothesis
Inductive Consequent
Want to prove this
Examples

\[ 1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \]

Generalized deMorgan’s Law

\[ n^3 - n \] is divisible by 3

Sum of powers of 2

\[ n! < n^n \]
Summation Notation

or Sigma notation:

\[ \sum_{i=1}^{n} a_i \] denotes the sum \( a_1 + a_2 + \ldots + a_n \)
Variations

\[ \sum_{i=1}^{n} a_i \]

\[ \sum_{j=1}^{4} f(j) \]

\[ \sum_{x \in \mathbb{N}} \frac{1}{x!} \]

\[ \sum_{i=0}^{2} a_i \]

\[ \sum_{k=1}^{3} (2k+1) \]

\[ \sum_{v \in V} \text{deg}(v) = 2e \]
Write the Summation Formula for ...

\[ 2 + 2 + 2 + 2 + 2 \]

\[ -1 + 1 + -1 + 1 + -1 + 1 \]

\[ 6 + 8 + 10 + 12 + 14 + 16 \]
Compute this ...

$$\sum_{i=2}^{4} i^2 =$$

$$\sum_{k=5}^{5} k^3 =$$

$$\sum_{i=1}^{2} \sum_{j=0}^{3} i \cdot j =$$
Some important summation formulas

\[ \sum_{i=1}^{n} 1 = 1 + 1 + ... + 1 = n \]

More generally, \[ \sum_{i=a}^{b} 1 = b - a + 1 \]

\[ \sum_{i=1}^{n} i = 1 + 2 + ... + n = n(n+1)/2 \]

\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6 \]

\[ \sum_{i=0}^{n} 2^i = 2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1 \]

More generally, \[ \sum_{i=0}^{n} ar^i = a(r^{n+1} - 1)/(r - 1) \quad (r \neq 1) \]
Basic summation rules

- $\sum c a_i = c \sum a_i$

- $\sum (a_i \pm b_i) = \sum a_i \pm \sum b_i$

- $\sum_{i=k}^{n} x_i = \sum_{i=k}^{m} x_i + \sum_{i=m+1}^{n} x_i$
Compute the sums

\[ \sum_{i=3}^{n+1} 1 = \]

\[ \sum_{i=2}^{n-2} i = \]

\[ \sum_{i=1}^{n-1} 2^i = \]

\[ 1 + 2 + 4 + \ldots + 100 = \]

\[ 2 + 4 + 8 + \ldots + 1024 = \]

\[ 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = \]

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Try this

• Consider the sum: $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

  – Write it in summation notation
  – Find a formula for the sum you wrote (consider using a picture?)
  – Prove your formula is correct using mathematical induction