Sets and Logic

Basic terminology

A set is an unordered collection of distinct objects called elements or members of the set.

The notation $x \in S$ — means “$x$ is an element of $S$”

Example: $S = \{2, 4, 6, 8\}$
$2 \in S$ — “$2$ is an element of $S$”
$3 \notin S$ — “$3$ is not an element of $S$”

Example: $S = \{\{2, 4\}, \{6\}, 8\}$, $|S| = 3$
$\{2, 4\} \in S$ — “$\{2, 4\}$ is an element of $S$”
$2 \notin S$ — “$2$ is not an element of $S$”

A multiset or a bag is an unordered collection of objects that are not necessarily distinct.

Sets and cardinality

The cardinality of a set $S$, denoted $|S|$, is the number of members of $S$.

Example: Let $A = \{a, b, c\}$, $B = \{1, 2\}$
$|A| = 3$ $|B| = 2$

Example: $S = \{2, 4, 6, 8\}$ $|S| =$

Example: $S = \{\{2, 4\}, \{6\}, 8\}$, $|S| =$
Some important sets

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ - the set of natural numbers
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - the set of integers
- $\mathbb{Z}_2 = \{0, 1\}$ - the binary digits
- $\mathbb{R}$ - the set of real numbers
- $\mathbb{Q} = \{x \mid x = p/q$ where $p, q \in \mathbb{Z}, q \neq 0\}$ - the set of rational numbers
- The empty (or null) set, denoted by $\emptyset$, or \{\}\.

Describing sets

Two ways to describe a set:

1. By listing elements, e.g., $S = \{2, 4, 6, 8\}$
2. By a property, e.g.,

   $$T = \{x \mid x \text{ is an even positive integer}\}$$
   $$E = \{x \in \mathbb{Z} \mid \frac{x}{2} \in \mathbb{Z}\}$$

Subsets

$S$ is a subset of $T$, denoted $S \subseteq T$, iff every element of $S$ is also an element of $T$.

Examples:
- $\{a, b\} \subseteq \{a, b, c\}$
- $\{a, b, c\} \subseteq \{a, b, c\}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $S \subseteq S$ (for every $S$)
- $\emptyset \subseteq S$ (for every $S$)

$S$ is a proper subset of $T$, denoted $S \subset T$, iff $S$ is a subset of $T$ but $S \neq T$.

Examples:
- $\{a, b\} \subset \{a, b, c\}$
- $\{b\} \subset \{a, b, c\}$
- what about $\emptyset \subset S$ ??

The Power Set

The power set of a set $S$ is the set of all subsets of $S$. The power set of $S$ is denoted by $P(S)$.

$$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$
$$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$
$$P(\{a\}) = \{\emptyset, \{a\}\}$$
$$P(\emptyset) = \{\emptyset\}$$

What can we say about $|P(S)|$?
Set equality

Two sets $S$ and $T$ are equal, denoted $S = T$, iff they have the same elements, i.e., for every $x$:
- if $x \in S$ then $x \in T$
- and if $x \in T$ then $x \in S$

In other words:
- $S = T$ iff $S \subseteq T$ and $T \subseteq S$

Proof technique: double inclusion

Examples:
- $\{a, b\} = \{b, a\}$
- $\{1, 2, 3\} = \{x \mid x \text{ is an integer and } 0 < x < 4\}$
- $\{2, 4, 6\} = \{x \mid x = 2 \times y, \text{ where } y \in \{1, 2, 3\}\}$

The Universal Set

Sets $S$ and $T$ are equal, denoted $S = T$, iff they have the same elements, i.e., for every $x$:
- if $x \in S$ then $x \in T$
- if $x \in T$ then $x \in S$

What does this even mean????

The Universal Set $U$

We usually think of sets as subsets of a universal set $U$.
- Example: $\{a, b\}$ and $\{b, d, e\} \Rightarrow U = \{a, b, c, d, e\}$
  (or maybe $U = \{a, b, c, d, e, f, g, \ldots\}$ - usually determined by context)

The complement of $S$, denoted $S'$, is the set of elements of $U$ that are not in $S$.
Example: $\{b, d, e\} = \{a, c\}$

The set difference, denoted $S - T$ (or $S \setminus T$), is the set of elements of $S$ that are NOT also in $T$.
Examples:
- $\{a, b, c, d, e\} - \{b, d, e\} = \{a, c\}$ (Note: $S' = U - S$)
- $\{b, c\} - \{a, b\} = \{c\}$
**Venn diagrams**

- **Disjoint sets** $S$ and $T$
  
  - $S \cap T = \emptyset$

- $S$ and $T$ are not disjoint

  - $S \cap T \neq \emptyset$

**Set Union and Intersection**

- $S \cup T = \{ x \mid x \in S \text{ or } x \in T \}$

- $S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$

- **Example:** Let $S = \{1,2,3,4\}$ and $T = \{2,3,5\}$. Then
  
  - $S \cup T = \{1,2,3,4,5\}$
  
  - $S \cap T = \{2,3\}$

**Set difference and complement**

- $S - T = \{ x \mid x \in S \text{ and } x \notin T \}$

- $\overline{S} = U - S$

- **Example:** Let $U = \mathbb{N}$
  
  - $S = \{ x \mid x \text{ is an integer greater than 6} \}$
  
  - $T = \{ x \mid x \text{ is an even positive integer} \}$

  Then
  
  - $S - T = \{ x \mid x \text{ is an odd integer greater than 6} \}$
  
  - $\overline{S} = \{ x \mid x \text{ is an integer less than or equal to 6} \}$

**Generalized unions and intersections**

- $S_1 \cap S_2 \cap \ldots \cap S_n$ denoted by $\bigcap_{i=1}^{n} S_i$

- $S_1 \cup S_2 \cup \ldots \cup S_n$ denoted by $\bigcup_{i=1}^{n} S_i$

- **Example:** Let $S_i = \{ i \}$.

  - $\bigcap_{i=1}^{n} S_i = \emptyset$

  - $\bigcup_{i=1}^{n} S_i = \mathbb{N}$
Set identities

- \( S \cup \emptyset = S \) (Identity laws)
- \( S \cap U = S \)
- \( S \cup U = U \) (Dominance laws)
- \( S \cap \emptyset = \emptyset \)
- \( S \cup S = S \) (Idempotent laws)
- \( S \cup (T \cup R) = (S \cup T) \cup R \) (Associative laws)
- \( S \cap (T \cap R) = (S \cap T) \cap R \) (Distributive laws)
- \( S \cup (T \cap R) = (S \cup T) \cap (S \cup R) \) (De Morgan’s laws)
- \( S \cap (T \cup R) = (S \cap T) \cup (S \cap R) \) (Inclusion-exclusion laws)
- \( (S) = S \) (Complement law)

Proving set identities - example

Prove that \( S \cap T = S \cup T \) (de Morgan’s Law for sets).

Proof: We proceed by showing that each set is a subset of the other, i.e. \( S \cap T \subseteq S \cup T \) and \( S \cup T \subseteq S \cap T \).

1. Suppose \( x \in S \cap T \), i.e. \( x \notin S \cup T \). Then \( x \notin S \) or \( x \notin T \). Hence, \( x \in S \) or \( x \in T \). This means that \( x \in S \cup T \).

Thus, \( S \cap T \subseteq S \cup T \).

2. Now suppose \( x \in S \cup T \). Then \( x \notin S \) or \( x \notin T \). Hence \( x \notin S \) and \( x \notin T \), which means that \( x \notin S \cap T \).

Therefore, \( S \cup T \subseteq S \cap T \).

Cartesian product

Let \( A = \{a, b, c\} \) and \( B = \{1, 2\} \).

The **cartesian product** is the set of ordered pairs \((x, y)\) where \( x \in A \) and \( y \in B \):

\[
A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}
\]

**Product Principle:** \( |A \times B| = |A| \cdot |B| \)

Ordered pairs and n-tuples

- **Ordered pairs** \((a_1, a_2)\)
- **Ordered n-tuples** \((a_1, a_2, \ldots, a_n)\)

- represent sequences where the order of elements does matter and repetitions are allowed.

The **Cartesian product** of the sets \( S_1, S_2, \ldots, S_n \) denoted by \( S_1 \times S_2 \times \ldots \times S_n \) is the set of all ordered \( n \)-tuples \((s_1, s_2, \ldots, s_n)\) where \( s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n \). In other words,

\[
S_1 \times S_2 \times \ldots \times S_n = \{(s_1, s_2, \ldots, s_n) \mid s_1 \in S_1 \text{ and } s_2 \in S_2 \text{ and } \ldots \text{ and } s_n \in S_n\}
\]
**Partitions**

A *partition* of a set $X$ is a collection of disjoint nonempty subsets of $X$ that have $X$ as their union.

The collection $X_1, X_2, X_3, X_4$ is a partition of $X$.

**Cardinality of disjoint set unions**

Let $A = \{a, b, c\}$, $B = \{1, 2\}$

*cardinality* of a set = number of members

$|A| = 3$

$|B| = 2$

$A \cup B = \{a, b, c, 1, 2\}$  
$A \cap B = \emptyset$

**Sum Principle:** If $A$ and $B$ are *disjoint*  
$|A \cup B| = |A| + |B|$

**Partitions - Example**

Let $X = A \cup B \cup C \cup D$ where  
$A = \{0, 4, 8\}$, $B = \{1, 5\}$, $C = \{2, 6\}$, $D = \{3, 7\}$

The sets $A, B, C, D$ form a partition of $X$.

**Cardinality of subset set difference**

Let $A = \{a, b, c, d, e\}$, $B = \{b, d\}$

$|A| = 5$

$|B| = 2$

$A - B = \{a, c, e\}$

**Difference Principle:** If $B \subseteq A$,  
$|A - B| = |A| - |B|$