Sets and Logic

CSC 1300 – Discrete Structures
Villanova University
Major Themes

• Sets
  – Ways of defining sets
  – Subsets, complements, the universal set
  – Venn diagrams
  – Set identities
  – Cartesian product
  – Partitions
Basic terminology

A **set** is an *unordered* collection of *distinct* objects called *elements* or *members* of the set.

The notation $x \in S$ — means “$x$ is an element of $S$”

**Example:** $S = \{2, 4, 6, 8\}$
- $2 \in S$ — “$2$ is an element of $S$”
- $3 \notin S$ — “$3$ is not an element of $S$”

**Example:** $S = \{\{2, 4\}, \{6\}, 8\}$, $|S| = 3$
- $\{2, 4\} \in S$ — “$\{2,4\}$ is an element of $S$”
- $2 \notin S$ — “$2$ is not an element of $S$”

A **multiset** or a **bag** is an *unordered* collection of objects that are not necessarily distinct.
Sets and cardinality

The **cardinality** of a set $S$, denoted $|S|$, is the number of members of $S$.

**Example:** Let $A = \{a, b, c\}$, $B = \{1, 2\}$

$|A| = 3 \quad |B| = 2$

**Example:** $S = \{2, 4, 6, 8\}$ \hspace{1cm} $|S| =$

**Example:** $S = \{\{2, 4\}, \{6\}, 8\}$ \hspace{1cm} $|S| =$
Some important sets

• \( \mathbb{N} = \{ 1, 2, 3, \ldots \} \) - the set of **natural numbers**
• \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) - the set of **integers**
• \( \mathbb{Z}_2 = \{0, 1\} \) - the **binary digits**
• \( \mathbb{R} \) - the set of **real numbers**
• \( \mathbb{Q} = \{ x \mid x = p/q \text{ where } p, q \in \mathbb{Z}, q \neq 0 \} \) - the set of **rational numbers**
• The **empty** (or **null**) **set**, denoted by \( \emptyset \), or \{ \}. 
Describing sets

Two ways to describe a set:

1. By listing elements, e.g., \( S = \{2, 4, 6, 8\} \)

2. By a property, e.g.,

\[
T = \{x \mid x \text{ is an even positive integer}\}
\]

\[
E = \{x \in \mathbb{Z} \mid \frac{x}{2} \in \mathbb{Z}\}
\]
Subsets

$S$ is a **subset** of $T$, denoted $S \subseteq T$, iff every element of $S$ is also an element of $T$.

**Examples:**
- $\{a, b\} \subseteq \{a, b, c\}$
- $\{a, b, c\} \subseteq \{a, b, c\}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $S \subseteq S$ (for every $S$)
- $\emptyset \subseteq S$ (for every $S$)

$S$ is a **proper subset** of $T$, denoted $S \subset T$, iff $S$ is a subset of $T$ but $S \neq T$.

**Examples:**
- $\{a, b\} \subset \{a,b,c\}$
- $\{b\} \subset \{a, b, c\}$
- what about $\emptyset \subset S$ ???
The **power set** of a set $S$ is the set of all subsets of $S$. The power set of $S$ is denoted by $P(S)$.

- $P(\emptyset) = \{\emptyset\}$
- $P(\{a\}) = \{\emptyset, \{a\}\}$
- $P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

What can we say about $|P(S)|$?
Set equality

Two sets \( S \) and \( T \) are \textit{equal}, denoted \( S = T \), iff they have the same elements, i.e., for every \( x \):

- if \( x \in S \) then \( x \in T \)
- \( \text{and} \) if \( x \in T \) then \( x \in S \)

In other words:

\[ S = T \iff S \subseteq T \text{ and } T \subseteq S \]

Proof technique: double inclusion
Set equality

Examples:

• \{a,b\} = \{b,a\}

• \{1, 2, 3\} = \{x \mid x \text{ is an integer and } 0 < x < 4\}

• \{2, 4, 6\} = \{ x \mid x = 2*y, \text{ where } y \in \{1, 2, 3\} \}
Sets $S$ and $T$ are equal, denoted $S = T$, iff they have the same elements, i.e., for every $x$:

- If $x \in S$ then $x \in T$
- If $x \in T$ then $x \in S$

What does this even mean????
The Universal Set $U$

We usually think of sets as subsets of a **universal set** $U$.

- **Example:** $\{a,b\}$ and $\{b,d,e\} \Rightarrow U = \{a,b,c,d,e\}$
  
  (or maybe $U = \{a,b,c,d,e,f,g,...,z\}$ - usually determined by context)

The **complement** of $S$, denoted $\overline{S}$ is the set of elements of $U$ that are not in $S$.

**Example:** $\{b,d,e\} = \{a,c\}$

The **set difference**, denoted $S - T$ (or $S \setminus T$), is the set of elements of $S$ that are NOT also in $T$.

**Examples:** $\{a,b,c,d,e\} - \{b,d,e\} = \{a,c\}$  
(Note: $\overline{S} = U - S$)  
$\{b,c\} - \{a,b\} = \{c\}$

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Venn diagrams

- Disjoint sets $S$ and $T$
- $S$ and $T$ are not disjoint
- $S \subseteq T$

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Set Union and Intersection

\[ S \cup T = \{ x \mid x \in S \text{ or } x \in T \} \]

\[ S \cap T = \{ x \mid x \in S \text{ and } x \in T \} \]

Example: Let \( S = \{1,2,3,4\} \) and \( T = \{2,3,5\} \). Then

\[ S \cup T = \{1,2,3,4,5\} \]

\[ S \cap T = \{2,3\} \]

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Set difference and complement

\[ S - T = \{ x \mid x \in S \text{ and } x \notin T \} \]

\[ \overline{S} = U - S \]

Example: Let \( U = \mathbb{N} \)

\[ S = \{ x \mid x \text{ is an integer greater than 6} \} \]
\[ T = \{ x \mid x \text{ is an even positive integer} \} \]

Then \( S - T = \{ x \mid x \text{ is an odd integer greater than 6} \} \)
\( \overline{S} = \{ x \mid x \text{ is an integer less than or equal to 6} \} \)

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Generalized unions and intersections

\[ S_1 \cap S_2 \cap \ldots \cap S_n \] denoted by \( \bigcap_{i=1}^{n} S_i \)

\[ S_1 \cup S_2 \cup \ldots \cup S_n \] denoted by \( \bigcup_{i=1}^{n} S_i \)

Example: Let \( S_i = \{ i \} \).

\[ \bigcap_{i=1}^{n} S_i = \quad \text{and} \quad \bigcup_{i=1}^{n} S_i = \]
## Set identities

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**Complementation law**

$(\overline{S}) = S$
Proving set identities - example

Prove that \( S \cap T = S \cup T \) (de Morgan’s Law for sets).

Proof: We proceed by showing that each set is a subset of the other, i.e. \( S \cap T \subseteq S \cup T \) and \( S \cup T \subseteq S \cap T \).

1. Suppose \( x \in S \cap T \). i.e. \( x \notin S \cap T \). Then \( x \notin S \) or \( x \notin T \).

   Hence, \( x \in S \) or \( x \in T \). This means that \( x \in S \cup T \).

   Thus, \( S \cap T \subseteq S \cup T \).

2. Now suppose \( x \in S \cup T \). Then \( x \notin S \) or \( x \notin T \).

   Hence \( x \notin S \) or \( x \notin T \), which means that \( x \notin S \cap T \).

   Therefore, \( x \in S \cap T \). Thus, \( S \cup T \subseteq S \cap T \).
Cartesian product

Let $A = \{a, b, c\}, \quad B = \{1, 2\}$

The **cartesian product** is the set of ordered pairs $(x, y)$ where $x \in A$ and $y \in B$:

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

**Product Principle:** $|A \times B| = |A| \cdot |B|$
Ordered pairs and n-tuples

**ordered pairs** $(a_1,a_2)$

and

**ordered n-tuples** $(a_1,a_2,...,a_n)$

• represent sequences where the order of elements **does** matter and repetitions are allowed.

The **Cartesian product** of the sets $S_1, S_2, ..., S_n$, denoted by $S_1 \times S_2 \times ... \times S_n$, is the set of all ordered $n$-tuples $(s_1,s_2,...,s_n)$ where $s_1 \in S_1$, $s_2 \in S_2$, ..., $s_n \in S_n$. In other words,

$$S_1 \times S_2 \times ... \times S_n = \{(s_1,s_2,...,s_n) \mid s_1 \in S_1 \text{ and } s_2 \in S_2 \text{ and } ... \text{ and } s_n \in S_n\}$$
Partitions

A *partition* of a set $X$ is a collection of disjoint nonempty subsets of $X$ that have $X$ as their union.

The collection $X_1, X_2, X_3, X_4$ is a partition of $X$. 

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Partitions - Example

Let $X = |A \cup B \cup C \cup D|$ where
$A = \{0, 4, 8\}, \ B = \{1, 5\}, \ C = \{2, 6\}, \ D = \{3, 7\}$

*The sets $A, B, C, D$ form a partition of $X*
Cardinality of disjoint set unions

Let \( A = \{a, b, c\} \), \( B = \{1, 2\} \)

**cardinality** of a set = number of members

\[
|A| = 3
\]

\[
|B| = 2
\]

\( A \cup B = \{a, b, c, 1, 2\} \quad A \cap B = \emptyset \)

**Sum Principle:** If \( A \) and \( B \) are disjoint

\[
|A \cup B| = |A| + |B|
\]
Cardinality of subset set difference

Let \( A = \{a, b, c, d, e\} \), \( B = \{b, d\} \)

\[ |A| = 5 \]
\[ |B| = 2 \]

\( A - B = \{a, c, e\} \)

**Difference Principle:** If \( B \subseteq A \),

\[ |A - B| = |A| - |B| \]