Graph Traversals

Major Themes

Graph traversals:
- Euler circuit/path
  - Every edge exactly once
- Hamilton circuit/path
  - Every vertex exactly once
  - Shortest circuit:
    - Traveling Salesman Problem
    - Shortest Path:
      - Dijkstra’s algorithm

The Bridges of Königsberg Puzzle

18th century map of the city of Königsberg with 7 bridges over the Pregel river

Find a walk through the city that would cross each bridge once and only once before returning to the starting point

Euler’s Solution to The Bridges of Königsberg

Leonhard Euler (1707-1783) rephrased the question in terms of the multigraph (i.e., multiple edges allowed)

Is there a cycle in the multigraph that traverses all its edges exactly once?
Euler cycles (circuits) and paths

Euler cycle: a cycle traversing all the edges of the graph exactly once
Euler path: a path (not a cycle) traversing all the edges of the graph exactly once

Examples:

- Has an Euler cycle
- Has neither Euler cycle nor Euler path
- Has an Euler path, but no Euler cycle

Necessary and sufficient conditions for Euler cycles

Theorem 12.3.1 (1) A connected graph G has an Euler cycle if and only if every vertex of G has even degree.

\(\Rightarrow\) A necessary condition

G has an Euler cycle only if every vertex has even degree

Assume G has an Euler cycle. Observe that every time the cycle passes through a vertex, it contributes 2 to the vertex’s degree, since the cycle enters via an edge incident with this vertex and leaves via another such edge.

\(\Leftarrow\) A sufficient condition

If every vertex in G has even degree, G has an Euler cycle

Lemma Assume every vertex in a multigraph has even degree. Start at an arbitrary non-isolated vertex \(v_0\), choose an arbitrary edge \((v_0, v_1)\), then choose an arbitrary unused edge from \(v_1\) and so on. Then after a finite number of steps the process will arrive at the starting vertex \(v_0\), yielding a cycle with distinct edges.

Proof In the above procedure, once you entered a vertex \(v\), there will always be another unused edge to exit \(v\) because \(v\) has an even degree and only an even number of the edges incident with it had been used before you entered it. The only edge from which you may not be able to exit after entering it is \(v_0\) (because an odd number of edges incident with \(v_0\) have been used as you didn’t enter it at the beginning), but if you have reached \(v_0\), then you have already constructed a required cycle.

A procedure for constructing an Euler cycle

Algorithm Euler(G)

//Input: Connected graph G with all vertices having even degrees
//Output: Euler cycle

Construct a cycle in G using the procedure from Lemma
Remove all the edges of cycle from G to get subgraph H
while H has edges

- find a non-isolated vertex v that is both in cycle and in H
- //the existence of such a vertex is guaranteed by G’s connectivity
- construct subcycle in H using Lemma’s procedure
- splice subcycle into cycle at v
- remove all the edges of subcycle from H

return cycle
Necessary and sufficient conditions for Euler paths

**Theorem 12.3.1 (2)** A connected graph $G$ has an Euler path but not an Euler cycle if and only if it has exactly two vertices of odd degree.

**Icosian Game**

Icosian Game This puzzle has been invented by the renowned Irish mathematician Sir William Hamilton (1805-1865) and presented to the world under the name of the Icosian Game. The game’s board was a wooden board with holes representing major world cities with grooves representing connections between them. The object of the game was to find a circular route that would pass through all the cities exactly once before returning to the starting point.
Hamilton Circuit and/or Traversal?

- $C_3$, $K_4$, $G$

- $K_6$, $G$

Hamilton Circuits: **Sufficient condition**

- Let $G$ be a graph with $n$ vertices where $n \geq 3$. If the degree of every vertex is more than $n/2$, then $G$ has a Hamilton circuit.

- Note: **NOT a necessary condition**

- No property is known to efficiently verify existence of a Hamilton cycle/path for general graphs. Moreover, the problem is known to be as difficult as the TSP

**Traveling Salesman Problem**

Find the lowest cost Hamilton circuit

Optimal tour of the 13,509 cities in the USA with populations greater than 500.

**Shortest Paths**

Dijkstra’s Algorithm:

- Build a tree of minimal length paths from root to leaves
- Similar to Prim’s MST algorithm