Graph Planarity

CSC 1300 – Discrete Structures
Villanova University

Three house/three utility puzzle

- Is it possible to join three houses to three separate utilities so that none of the connections cross?

Planar graphs

A graph is **planar** if it can be drawn in the plane without any edges crossing.

Major Themes

- Planar and non-planar graphs
- The graphs $K_5$ and $K_{3,3}$
- Dividing the plane into regions
- Euler’s formula
More examples
Geogebra files available at http://www.toroidalsnark.net/dmwlinksfiles.html

K₃,₃ is not planar

- Consider region R formed by vertices 1, 2, 4, 5.
- **Case 1:** vertex 3 **inside** R
- **Case 2:** vertex 3 **outside** R

Where can vertex 6 go?

K₅ is not planar
Prove using a similar argument based on regions.

Kuratowski’s Theorem
Are there non-planar graphs other than K₃,₃ and K₅?

*What makes a graph non-planar?*
- If a graph has K₃,₃ or K₅ as a subgraph
- If a graph has a subgraph that can be obtained by adding “intermediate” vertices to edges of either K₃,₃ or K₅ (homeomorphic)
- any others?

Kuratowski’s Theorem:
A graph is nonplanar if and only if it has a subgraph homeomorphic to either K₃,₃ or K₅.
Previous example – non-planar

Here’s a problem: contains $K_{3,3}$ subgraph!

Regions of the plane

- Planar graphs divide the plane into regions called \textit{faces}

Example: Relating number of faces to vertices and edges

How many edges (walls) can we remove before all the fields are flooded? What type of graph are we left with after all these walls are removed?

Example:

- Number of faces:
- Number of vertices:
- Number of edges:
  - edges removed in flood:
  - edges remaining:

Figure 11.3. A map of the fields on Iki Island, with watch platforms marked as small circles.
**Euler’s formula for planar graphs**

**Theorem:** Let $G$ be a connected planar graph with $v$ vertices, $e$ edges and $f$ faces. Then: $v - e + f = 2$

**Corollary:** If $G$ is connected planar graph, then $e \leq 3v - 6$

Intuition: graphs with lots of edges cannot be planar!

**Example:** What can we say about $K_5$? $K_{3,3}$?

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**Previous example**

$e = \_\_\_$

$v = \_\_\_\_$

$e > 3v - 6$ ?

Planar?