Binomial Coefficients and Pascal’s triangle

CSC 1300 – Discrete Structures
Villanova University
Permutations

• A permutation is an ordering of objects
  – For example, 3 blocks can be ordered 6 ways

• There are n! permutations of n elements
  – Easily proved using the Product rule
k-Combination

What if all that matters is which blocks you select, not the order?...

• A **combination** is an *unordered* arrangement of elements in a set
  – Example: a 3-combination from a set of 12 colored blocks
Combinations

- **Choice notation:** \( n \text{ choose } k \)
  \[
  \binom{n}{k} = \text{ } k\text{-combinations of a set with } n \text{ elements}
  \]
- also denoted: \( C(n,k) \)
- The number of subsets of size \( k \) from a set with \( n \) elements
- Also called the *binomial coefficient*

- Formula: \( C(n, k) = \binom{n}{k} = \frac{n!}{(n - k)! \cdot k!} \)
Combinations

• **Example:** the number of ways to form a committee of 4 members from a department of 13 faculty

• denoted $C(13,4)$ or $\binom{13}{4}$
k-Permutations

\[ P(n,k) = k\text{-permutations of a set with } n \text{ elements} \]

- The number of ways to permute \( k \) out of \( n \) items
- Similar to combinations, but order matters

- Formula: \[ P(n, k) = \frac{n!}{(n - k)!} \]
Permutations

• **Example:** the number of ways to choose 4 of the 13 faculty to teach special topics courses.

• denoted $P(13,4)$
Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
Pascal’s Triangle
Pascal’s Triangle

\[
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \binom{1}{1} \\
\binom{2}{0} \binom{2}{1} \binom{2}{2} \\
\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}
\end{array}
\]
Pascal’s Triangle

- Pascal’s Triangle represents the identity:

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

for \(0 \leq k \leq n\)
More Properties of Combinations

\[ C(n, k) = C(n, n-k) \quad \text{for any } 0 \leq k \leq n \]

\[ C(n, 0) = \quad \text{for any } n \geq 0 \]

\[ C(n, n) = \]

\[ C(n, 0) + C(n, 1) + \ldots + C(n, n) = \quad \text{for any } n \geq 0 \]
Binomial Refresher

• A binomial expression is simply the sum of two terms
  – For example:
    • (x+y)
    • (x+y)^2

• When a binomial expression is expanded, the binomial coefficients can be “seen”
  – For example:
    
    \[(x+y)^2 = x^2 + 2xy + y^2\]
    
    \[= 1x^2 + 2xy + 1y^2\]
Binomial Coefficients & Combinations

• Explore the following:

\[(x+y)^3 = (x+y)(x+y)(x+y)\]

\[xxx + xxy + yxx + yyy + yxx + yxy + yyy\]

\[x^3 + 3x^2y + 3xy^2 + y^3\]

\[C(3,0) \quad C(3,1) \quad C(3,2) \quad C(3,3)\]

• Binomial Theorem

\[(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k\]
Binomial Theorem

• Problem
  – What is the expansion of $(x+y)^4$?

  – Find the coefficient $x^4y^7$ in the expansion of $(2x+y)^{11}$
Some Corollaries of the Binomial Theorem

Corollary 1 \((a = b = 1)\):

Corollary 2 \((a = 1, b = -1)\):

Corollary 3 \((a = 1, b = 2)\):
Bit strings representing sets

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