Counting Principles

CSC 1300 – Discrete Structures
Villanova University
Counting: Review from Chapter 1

• Product Principle: \[ |A \times B \times \ldots \times N| = |A| \cdot |B| \cdot \ldots \cdot |N| \]

• Sum Principle: \[ |A \cup B \cup \ldots \cup N| = |A| + |B| + \ldots + |N| \]
  \[\text{(for disjoint sets only)}\]

• Pigeonhole Principle: \( n \) pigeonholes ..... \( p \) pigeons
  – \( p > n \) pigeons \( \Rightarrow \) at least one pigeonhole has more than one pigeon
  – \( p > kn \) \( \Rightarrow \) at least one hole with more than \( k \) pigeons.
Permutations and Combinations – Review from Chapter 6

\[ P(n,k) = \text{k-permutations of a set with n elements} \]
- order matters
- Formula: \( P(n,k) = \frac{n!}{(n-k)!} \)

\[ C(n,k) = \text{k-combinations of a set with n elements} \]
- order does NOT matter
- Formula: \( C(n,k) = \frac{n!}{[(n-k)!k!]} \)

\[ \text{divide by } k! \text{ due to overcounting} \]
Product Principle

• Cardinality of cartesian product of sets
• Choose an element from **each** of several sets

**Product Principle:** \(|A \times B \times C| = |A| \cdot |B| \cdot |C|\)
Applying the Product Principle

• If an activity can be constructed of successive steps, to determine the possibilities,
  – Multiply together the number of ways of doing each step

• In other words, if
  
  Step 1 = \( n_1 \) ways
  Step 2 = \( n_2 \) ways
  ....
  Step \( t \) = \( n_t \) ways

  Then the number of possibilities = \( n_1 * n_2 * n_3 * \ldots n_t \)
Product Principle – simple example

• **Problem**
  – How many different license plates are available if each plate contains a sequence of 3 letters followed by three digits?

• **Solution**
  – 26 choices for each letter and 10 choices for each number

  \[
  26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000
  \]
  possible license plates
Some Questions from Section 7.2

• **Question A.** How many ways are there to place \( k \) differently labeled balls, at most one per box, into \( n \) labeled boxes?

• **Question B.** How many ways are there to place \( k \) identical (unlabeled) balls, at most one per box, into \( n \) labeled boxes?

• **Question C.** How many ways are there to place balls, exactly one per box, with \( k \) different possible labels, into \( n \) labeled boxes?
Sum Principle

- Cardinality of disjoint union of sets
- Choose an element from **one** of several sets

**Sum Principle:** \(|A \cup B \cup C| = |A| + |B| + |C|\)

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Applying the Sum principle – simple example

• **Problem**
  – Your broker has told you to select a stock from one of the following lists:
    • 25 high tech companies
    • 15 consumer product companies
    • 10 service companies
  – How many choices do you have?

• **Solution**
  – $25 + 15 + 10 = 50$ choices

**Note:** Must have disjoint sets of objects
Using Sum & Product Principles

• **Problem**
  – How many strings are there of lowercase alpha characters of length four or less?

• **Solution**
  – There are 26 lowercase alpha characters
    • For 4 characters, there are \(26 \times 26 \times 26 \times 26 = 456,976\) possible strings
    • For 3 characters, there are \(26 \times 26 \times 26 = 17,576\) possible strings
    • For 2 characters, there are \(26 \times 26 = 676\) possible strings
    • For 1 character, there are 26 possible strings
    • Thus the total \(456,976 + 17,576 + 676 + 26 = 475,254\)
    • Let’s not forget the empty string so we have a total of 475,255 possible strings
Using Sum & Product Principles

• Problem
  How many bit strings of length 5 begin with 00 or with 11?

• Solution
  There are $2^3$ five bit strings that begin with 00
  There are $2^3$ five bit strings that begin with 11
  Therefore, there are a total of $2^3 + 2^3$ eight bit strings that begin with 00 or with 11.
Principle of Inclusion-Exclusion

- Sometimes, it is not possible to apply the sum principle because the sets are not disjoint, so we overcount elements in their intersection.

\[ |A_1 \cup A_2 | = |A_1| + |A_2| - |A_1 \cap A_2| \]

Compare with sum rule (when \( |A_1 \cap A_2| = \emptyset \)): \[ |A_1 \cup A_2 | = |A_1| + |A_2| \]
Principle of Inclusion-Exclusion

• Problem
  How many bit strings of length 5 begin with 0 or end with 11?

• Solution
  There are $2^4$ five bit strings that begin with 0
  There are $2^3$ five bit strings that end with 11
  There are $2^2$ five bit strings that begin with 0 and end with 11 (overcount)
  Therefore, there are a total of $2^4+2^3-2^2$ eight bit strings that begin with 0 and end with 11.
Principle of Inclusion-Exclusion

• Problem
  How many bit strings of length eight either begin with 111 or end with 00?
Principle of Inclusion-Exclusion

7.5 Check Yourself #2:
How many permutations of the numbers 0,1,2,3,4 have substrings 03 or 21?
Principle of Inclusion-Exclusion

Example 7.5.4
How many permutations of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 have the substrings 53, 02, or 28?
(e.g., the permutation 4536028179 has all three, but 8042957316 has none.)

7.5 Check Yourself #1
Write out Theorem 7.5.3 for four sets $A_1, A_2, A_3, A_4$

• Generalized Inclusion/Exclusion: Theorem 7.5.3