Binomial Coefficients and Pascal’s triangle

CSC 1300 – Discrete Structures

Villanova University
Permutations

• A permutation is an ordering of objects
  – For example, 3 blocks can be ordered 6 ways

• There are n! permutations of n elements
  – Easily proved using the Product rule
k-Combination

What if all that matters is which blocks you select, not the order?...

• A combination is an unordered arrangement of elements in a set
  – Example: a 3-combination from a set of 12 colored blocks
Combinations

- **Choice notation:** \( n \text{ choose } k \)
  \[
  \binom{n}{k} = k\text{-combinations of a set with } n \text{ elements}
  \]

- also denoted: \( C(n,k) \)

- The number of subsets of size \( k \) from a set with \( n \) elements

- Also called the *binomial coefficient*

- Formula: \( C(n,k) = \frac{n!}{(n-k)!k!} \)
Combinations

• **Example:** the number of ways to form a committee of 4 members from a department of 13 faculty

• denoted \( C(13, 4) \) or \( \binom{13}{4} \)
k-Permutations

\[ P(n,k) = k\text{-permutations of a set with } n \text{ elements} \]

- The number of ways to permute \( k \) out of \( n \) items
- Similar to combinations, but order matters
- Formula: \( P(n,k) = n! / (n-k)! \)
Permutations

• **Example:** the number of ways to arrange 4 of the 13 faculty to appear in a photo

• denoted $P(13,4)$
Pascal’s Triangle

• Pascal’s Triangle represents the identity:
  \[ C(n+1,k) = C(n,k) + C(n,k-1) \]
  for any \( 1 \leq k \leq n \)
More Properties of Combinations

\[ C(n, k) = C(n, n-k) \quad \text{for any } 0 \leq k \leq n \]

\[ C(n, 0) = \quad \text{for any } n \geq 0 \]

\[ C(n, n) = \]

\[ C(n, 0) + C(n, 1) + \ldots + C(n, n) = \quad \text{for any } n \geq 0 \]
Binomial Refresher

• A binomial expression is simply the sum of two terms
  – For example:
    • (x+y)
    • (x+y)^2

• When a binomial expression is expanded, the binomial coefficients can be “seen”
  – For example:
    (x+y)^2 = x^2 + 2xy + y^2
    = 1x^2 + 2xy + 1y^2
Binomial Coefficients & Combinations

• Explore the following:
  
  \[(x+y)^3 = (x+y)(x+y)(x+y)\]
  
  \[xxx + xxy + xyx + yyy + yxx + yxy + yyy + yyy\]
  
  \[x^3 + 3x^2y + 3xy^2 + y^3\]
  
  \[C(3,3) \quad C(3,2) \quad C(3,1) \quad C(3,0)\]

• Binomial Theorem
  
  \[(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k\]
Binomial Theorem

• **Problem**
  – What is the expansion of \((x+y)^4\)?

• **Solution**
  – \((x+y)^4 = C(4,0)x^4y^0 + C(4,1)x^3y^1 + C(4,2)x^2y^2 + C(4,3)x^1y^3 + C(4,4)x^0y^4\)

  \[= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
Binomial Coefficients & Combinations

• **Problem**
  
  – Find the coefficient $x^4y^7$ in the expansion of $(x+y)^{11}$

• **Solution**

  n = 11 and k = 7

  $C(11,7) x^{11-7} y^7$

  $(11*10*9*8*7*6*5)/(7*6*5*4*3*2) x^{11-7} y^7$

  $= 330 x^4 y^7$
Some Corollaries of the Binomial Theorem

Corollary 1 \((a = b = 1)\):

Corollary 2 \((a = 1, b = -1)\):

Corollary 3 \((a = 1, b = 2)\):
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bit strings to cut out for playing around with patterns

Villanova CSC 1300 - Dr Papalaskari
Below are rows zero to sixteen of Pascal's triangle in table form (even numbers highlighted): [source: wikipedia:

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