Major Themes

- Proof techniques
- Induction
- Summations

Types of proofs

- direct
- indirect (by contrapositive)
- by contradiction
- proof of equivalence
- proof by cases
- proof by mathematical induction

This week’s focus

Review

Proofs, examples, and counterexamples: \( \forall x P(x) \)

**For universal statements:**

- Checking validity of a theorem for specific examples does NOT constitute a proof (unless the examples exhaust all the values in the theorem’s domain, which is impossible if the latter is infinite).

- Just a single example suffices to disprove a theorem. (Such an example is usually called a counterexample).
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Mathematical Induction

- Use to prove universal statements
  \( \forall n P(n) \)

- Easy to think of it as the domino effect

Let \( P(n) \) be the proposition: \textit{domino } n \textit{is knocked over}

- If the first domino is knocked over (\( P(1) \) is true); and
- if whenever the \( n \text{th} \) domino is knocked over, then the \( (n+1) \text{th} \) is also knocked over
  \( \text{ (i.e., } P(n) \rightarrow P(n+1) \text{ is true}) \)

..... \textit{It follows that all the dominoes are knocked over.} \text{ (i.e., } \forall n P(n) \text{ is true) } \)

Examples

- \( P(n) = \text{“A set with cardinality } n \text{ has } 2^n \text{ subsets”} \) \text{ 4.2.4}
- \( P(n) = \text{Generalized deMorgan’s Law} \) \text{ 4.2.2}
- \( P(n) = \text{“A tree with } n \text{ vertices has } n \text{ edges”} \) \text{ 4.2.5}

Major Themes

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Summation Notation

Because induction is often used to prove facts about addition of sequences of values, we need to know how to read and write summations, using the *sigma notation*:

\[ \sum_{i=1}^{n} a_i \]

denotes the sum \( a_1 + a_2 + \ldots + a_n \)

The lower and upper limits \( l \) and \( u \) can be any integers provided \( l \leq u \).

Sigma Notation Variations

\[ \sum_{j=1}^{n} a_j \]

\[ \sum_{j=0}^{2} a_j \]

\[ \sum_{j=1}^{4} f(j) \]

\[ \sum_{k=1}^{3} (2k+1) \]

\[ \sum_{x \in \mathbb{N}} \frac{1}{x!} \]

\[ \sum_{v \in V} \deg(v) = 2e \]

Write the Summation Formula for …

2 + 2 + 2 + 2 …

-1 + 1 -1 + 1 + 1 + 1 …

6 + 8 + 10 + 12 + 14 + 16 …

Compute this …

\[ \sum_{k=2}^{4} k^2 = \]

\[ \sum_{k=5}^{5} k^3 = \]

\[ \sum_{i=1}^{2} \sum_{j=0}^{3} ij = \]
Some important sums

\[ \sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1 = n \]

More generally, \[ \sum_{i=1}^{u} 1 = u - 1 + 1 \]

\[ \sum_{i=1}^{n} i = 1 + 2 + \ldots + n = n(n+1)/2 \]

\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6 \]

\[ \sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \]

More generally, \[ \sum_{i=0}^{n} ar^i = a(r^{n+1} - 1)/(r - 1) \ (r \neq 1) \]

Basic summation rules

- \( \Sigma c a_i = c \Sigma a_i \)
- \( \Sigma (a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i \)
- \[ \sum_{i=1}^{u} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{u} a_i \]
- \[ \sum_{i=0}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \]

Compute the sums

a) \[ 1 + 3 + 5 + \ldots + 999 \]
b) \[ 2 + 4 + 8 + \ldots + 1024 \]
c) \[ \sum_{i=3}^{i} 1 = n^i - 1 \]
d) \[ \sum_{i=2}^{n} 1 = n - 2 \]
e) \[ \sum_{i=0}^{n} (i+1) = \sum_{i=0}^{n} i + \sum_{i=0}^{n} 1 \]
f) \[ \sum_{i=1}^{n} 3^{i-1} = \sum_{i=1}^{n} 3^{i-1} \]
g) \[ \sum_{i=1}^{n} \sum_{j=1}^{i} j = \sum_{i=1}^{n} \sum_{j=1}^{i} j \]

Which Proof Method?

1. Begin with a direct proof approach
2. If this fails, try either
   - indirect / contrapositive approach
   - proof by contradiction
   - proof by cases
   - a combination...
   - ..... 
   - If all else fails try mathematical induction