1. (15 points) **Differentiate the following functions.**
(a) \( f(x) = \sec x \)
\[ f'(x) = \sec x \tan x \]
(b) \( f(x) = \sqrt{3x^2 - 4x} \)
\[ f'(x) = \frac{1}{2}(3x^2 - 4x)^{-\frac{1}{2}} \cdot (6x - 4) \] using the chain rule where \( h(x) = \sqrt{x} \) and \( g(x) = 3x^2 - 4x \).
(c) \( f(x) = \ln \sqrt{e^x + \sin x} \)
\[ f'(x) = \frac{1}{\sqrt{e^x + \sin x}} \cdot \frac{1}{2}(e^x + \sin x)^{-\frac{1}{2}} \cdot (e^x + \sin x) \] (using the fact that \( \frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)} \) where \( g(x) = \sqrt{e^x + \sin x} \) and we use the chain rule to find the derivative of \( g(x) \)).

2. (12 points) **Find \( \frac{dy}{dx} \) by implicit differentiation for** \( \cos(xy) + e^{x^2} = 3y^3 - 1 \).
\[-\sin(xy) \left( x \frac{dy}{dx} + y \right) + 2xe^{x^2} = 9y^2 \frac{dy}{dx} \]
\[-x \sin(xy) \frac{dy}{dx} - 9y^2 \frac{dy}{dx} = y \sin(xy) - 2xe^{x^2} \]
\[ \frac{dy}{dx} (-x \sin(xy) - 9y^2) = y \sin(xy) - 2xe^{x^2} \]
\[ \frac{dy}{dx} = \frac{y \sin(xy) - 2xe^{x^2}}{-x \sin(xy) - 9y^2}. \]

3. (14 points) A cone-shaped coffee filter of radius 6 cm and depth 10 cm contains water, which drips out through a hole at the bottom so the volume of the water in the filter decreases at a rate of 1.5 cm³/sec. How fast is the water level falling when the depth is 8 cm? (Hint: The ratio of the radius to the depth of the remaining water will always be \( \frac{6}{10} \).)
Known rate $\frac{dV}{dt} = -1.5$, unknown rate $\frac{dh}{dt}$. The equation relating these two is $V = \frac{1}{3} \pi r^2 h$. From the hint we know $\frac{r}{h} = \frac{6}{10}$ and so $r = \frac{3h}{5}$. Plugging this into the equation for $V$ we get $V = \frac{1}{3} \pi \left(\frac{3h}{5}\right)^2 h^3$.

We can clean up constants and get $C = \frac{3}{25} \pi h^3$ and taking an implicit derivative gives

$$\frac{dV}{dt} = \frac{3}{25} \pi 3h^2 \frac{dh}{dt}.$$ 

Plugging in our known rate and $h = 8$ we get

$$\frac{dh}{dt} = \frac{-1.5}{25} \pi 3 \cdot 8^2 = \frac{25}{384 \pi}.$$

4. (a) (8 points) Find the linear approximation of the function $f(x) = \frac{1}{\sqrt{1+x}}$ at $a = 0$.

$L(x) = f(a) + f'(a)(x-a)$ and $f(0) = 1$ while $f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}} = -\frac{1}{2} \cdot \frac{1}{(1+x)^{3/2}}$ so $f'(0) = -\frac{1}{2}$.

Putting together we get $L(x) = 1 - \frac{1}{2}x$.

(b) (4 points) Use your answer in part (a) to approximate the number $\frac{1}{\sqrt{0.9}}$.

We need to plug in $x = -0.1$ to $L(x)$ to get $1 - \frac{1}{2}(-0.1) = 1.05$.

5. (12 points) The number of people riding the subway daily from point $A$ to point $B$ is a function $f(x)$ of the fare $x$ cents. Suppose $f(235) = 4600$ and $f'(235) = -100$.

(a) Explain in a complete sentence the meaning of the statement $f(235) = 4600$.

If the fare is $2.35$ then 4600 people take the subway daily from point $A$ to point $B$.

(b) Explain in a complete sentence the meaning of the statement $f'(235) = -100$.

Any increase in the fare from $2.35$ decreases ridership by 100.

(c) When the fare is $2.35$ (235 cents) would the number of people riding the subway daily increase or decrease if the fare was increased? Why?

Since the derivative is negative, an increase in fare will result in a decrease in the number of people riding the subway.

6. (12 points) Use logarithmic differentiation to find the derivative of $f(x) = x^{1/x}$.

We rewrite the equation as $y = x^{1/x}$ and take a natural log of both sides: $\ln y = \frac{1}{x} \ln x$. Implicitly differentiating (using the product rule on the right) gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \cdot 1 + \left(-\frac{1}{x^2}\right) \ln x.$$ 

We multiply both sides by $y$ and then replace $y$ with $x^{1/x}$ to get

$$\frac{dy}{dx} = \left(\frac{1}{x^2} - \frac{1}{x^2} \ln x\right) \cdot x^{1/x}.$$
7. (15 points) In an animal hospital, 8 units of a sulfate solution were injected into a dog. After 50 minutes only 4 units remained in the dog. The amount of sulfate present in the dog after \( t \) minutes decreases at a rate proportional to the amount remaining.

(a) Find the formula for the amount of sulfate remaining in the dog after \( t \) minutes.

We need to solve for \( P_0 \) and \( k \) in the equation \( P(t) = P_0e^{kt} \). We know \( P_0 = 8 \) and \( P(50) = 4 \) so to solve for \( k \), \( 4 = 8e^{k \cdot 50} \) And so \( k = \ln \frac{1}{2}/50 \approx -0.01386 \). So we have the equation

\[
P(t) = 8e^{-0.01386t}. 
\]

(b) How much sulfate will be left in the dog after 2 hours?

This is asking us to plug 120 (2 hours in minutes) into the answer for (a). \( P(t) = 8e^{120 \cdot (-0.01386)} \approx 1.516 \) units.

(c) When will there be 1 unit left?

This is asking us to find \( t \) so that \( P(t) = 1 = 8e^{-0.01386t} \). We divide by 8 and take a natural log of both sides to get \( \ln \frac{1}{8} = -0.01386t \) and solving for \( t \) gives \( \ln \frac{1}{8}/(-0.01386) \approx 150 \) minutes.

8. (8 points) Use implicit differentiation to find the derivative of \( y = \cos^{-1} x \)

First we rewrite this in terms of an inverse as \( x = \cos y \). Then we implicitly differentiate to get

\[
1 = -\sin y \frac{dy}{dx}.
\]

Solving for the derivative gives:

\[
\frac{dy}{dx} = -\frac{1}{\sin y}.
\]

To get rid of the \( y \) in our answer we use \( x = \cos y \) to get the following triangle (using Pythagorean theorem to find the third length):

\[
\sin y \text{ is then } \sqrt{1 - x^2} \text{ and so } \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.
\]