A simple proof of arithmetical completeness for Π_1-conservativity logic

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Abstract

Hájek and Montagna proved that the modal propositional logic ILM is the logic of Π_1-conservativity over sound theories containing $I\Sigma_1$ (PA with induction restricted to Σ_1 formulas). I give a simpler proof of the same fact.

1 Introduction

By a "theory" we mean an effectively axiomatized theory whose language contains that of PA (arithmetic).

A theory T_2 is said to be Π_1-*conservative* over a theory T_1, if T_1 proves every Π_1-theorem of T_2. And T_2 is *interpretable* in T_1 if, intuitively, the language of T_2 can be translated into the language of T_1 in such a way that T_1 proves the translation of every theorem of T_2.

A theory is said to be *essentially reflexive*, if for any formula α it proves $Pr_{PC}(\lceil\alpha\rceil) \to \alpha$, where $\lceil\alpha\rceil$ is the code (Gödel number) of α and $Pr_{PC}(x)$ is the standard formalization of "x is the code of a formula provable in the classical predicate calculus".

It is known that PA is essentially reflexive, but no finitely axiomatizable reasonable theory, including $I\Sigma_1$ (PA with induction restricted to Σ_1 formulas), can be such. Indeed, suppose T is a sufficiently strong finitely axiomatized theory. Let then Ax be the conjunction of the universal quantifiers closures of its axioms. If T is essentially reflexive, then $T \vdash Pr_{PC}(\lceil\neg Ax\rceil) \to \neg Ax$, whence

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$T \vdash \neg Pr_{PC}(\ulcorner \neg Ax \urcorner)$, which means that T proves its own consistency and hence, by Gödel's Second Incompleteness Theorem, T is inconsistent.

According to a nice fact known as *Orey-Hájek characterization*, if given theories are essentially reflexive, one is interpretable in another if and only if one is Π_1-conservative over the other; moreover, this fact is provable in PA, so we can say that interpretability and Π_1-conservativity relations between essentially reflexive theories are "the same". However, this is not true for finitely axiomatized theories like $I\Sigma_1$.

De Jongh and Veltman [5] introduced the propositional modal logic ILM, whose language contains two modal operators: $\Box$ (unary) and $\rhd$ (binary). Berarducci [1] and Shavrukov [7], independently, proved that ILM is the logic of interpretability over PA, that is, ILM yields exactly the schemata of PA-provable formulas, when $\Box A$ is understood as a formalization of "A is PA-provable" and $A \rhd B$ as a formalization of "$PA+B$ is interpretable in $PA+A$". By the Orey-Hájek characterization, this result immediately implies that ILM is the logic of Π_1-conservativity over PA as well. However, the question whether ILM is the logic of Π_1-conservativity over $I\Sigma_1$ (whose logic of interpretability was in [10] shown to be different from ILM) remained open until Hájek and Montagna [6] found a positive answer.

In this paper I present an alternative proof of completeness of ILM as the logic of Π_1-conservativity over $I\Sigma_1$ and its sound extensions; this proof is more direct[1] and therefore considerably simpler than that of Hájek and Montagna; since, in view of the Orey-Hájek characterization this result immediately implies completeness of ILM as the logic of interpretability over PA, this is at the same time a new proof of the above-mentioned Berarducci-Shavrukov theorem, which seems the simplest among those known so far.

2 Modal logic preliminaries

ILM is given as the classical propositional logic plus the rule of necessitation $\vdash A \Rightarrow \vdash \Box A$ and the following axiom schemata ($\Diamond = \neg\Box\neg$):

$$\Box(A \to B) \to (\Box A \to \Box B);$$
$$\Box(\Box A \to A) \to \Box A;$$
$$\Box(A \to B) \to (A \rhd B);$$
$$((A \rhd B) \wedge (B \rhd C)) \to (A \rhd C);$$
$$((A \rhd C) \wedge (B \rhd C)) \to ((A \vee B) \rhd C);$$
$$(A \rhd B) \to (\Diamond A \to \Diamond B);$$
$$(\Diamond A) \rhd A;$$
$$(A \rhd B) \to ((A \wedge \Box C) \rhd (B \wedge \Box C)).$$

[1] as it appeals only to the most elementary facts about Π_1-sentences and is based directly on the natural semantics for ILM, — Veltman models.

Thus, ILM contains the provability logic GL and, therefore, $ILM \vdash \Box A \to \Box\Box A$ (see [2]).

One can show that $ILM \vdash \Box A \leftrightarrow (\neg A) \rhd \bot$, which means that $\Box$ can be eliminated from the language of ILM.

A finite *Veltman frame* is a system $\langle W, R, \{S_w\}_{w\in W}\rangle$, where W is a finite nonempty set (of "worlds") and R and each S_w are binary relations on W such that the following holds:

1. R is transitive and irreflexive;
2. each S_w is transitive and reflexive;
3. uS_wv only if wRu and wRv;
4. $wRuRv \Longrightarrow uS_wv$;
5. $uS_wvRr \Longrightarrow uRr$.

A finite *Veltman model* is a system

$$\langle W, R, \{S_w\}_{w\in W}, \models\rangle,$$

where $\langle W, R, \{S_w\}_{w\in W}\rangle$ is a finite Veltman frame and $\models$ is a ("forcing") relation between worlds and ILM-formulas such that:

- The Boolean connectives are treated in the classical way: $w \not\models \bot$, $w \models A \to B \Longleftrightarrow (w \not\models A$ or $w \models B)$, etc.;
- $w \models \Box A \Longleftrightarrow$ (for all u, if wRu, then $u \models A$);
- $w \models A \rhd B \Longleftrightarrow$ (for all u, if wRu and $u \models A$, then there is v such that uS_wv and $v \models B$).

A formula A is said to be *valid* in a Veltman model $\langle W, R, \{S_w\}_{w\in W}, \models\rangle$, if $w \models A$ for all $w \in W$.

Theorem 1 (De Jongh and Veltman [5]) *$ILM \vdash A$ iff A is valid in all finite Veltman models.*

3 Arithmetic preliminaries

We fix a theory T containing $I\Sigma_1$. For safety we assume that T is in the language of arithmetic and T is sound, i.e. all its axioms are true (in the standard model of arithmetic).[2]

[2] In fact it is easy to adjust our proof of the completeness theorem to the weaker condition of Σ_1-soundness of T.

A *realization* is a function * which assigns an arithmetical sentence p^* to each propositional letter p of the modal language and which is extended to other modal formulas in the following way:

- * commutes with the Boolean connectives: $\bot^* = \bot$, $A \to B = A^* \to B^*$, etc.;
- $(\Box A)^* = Pr(\ulcorner A^* \urcorner)$;
- $(A \rhd B)^* = Conserv(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$,

where $Pr(\ulcorner A^* \urcorner)$ and $Conserv(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$ are natural formalizations of "A^* is T-provable" and "$T + B^*$ is Π_1-conservative over $T + A^*$".

We need to introduce some more notation and terminology.

We will write $\vdash_x F$ to say that x is the code of some T-proof of the formula F.

"Σ_1!" denotes the class of the arithmetical formulas which have an explicit Σ_1 form, i.e. $\exists x F$ for some primitive recursive formula F. And simply "Σ_1" denotes the class of the formulas which are T-provably equivalent to some Σ_1!-formula. Similarly for Π_1.

Let us fix $\exists y Regwitness(x, y)$ as a natural Σ_1!-formalization of the predicate "x is the code of a true Σ_1!-sentence", such that (T proves that) for each Σ_1!-sentence F, $T \vdash F \leftrightarrow \exists y Regwitness(\ulcorner F \urcorner, y)$.

Existence of the formula $Regwitness(x, y)$ is the only not very trivial, — but quite well known (see, e.g., [8]), — fact about Σ_1- (Π_1-) sentences that will be used in the arithmetical completeness proof below.

Now, we say that a natural number k is a *regular counterwitness* for a Π_1!-sentence $\forall x F$, if $Regwitness(\ulcorner \exists x \neg F \urcorner, \bar{k})$ is true.

4 The completeness theorem

Theorem 2 *$ILM \vdash A$ iff for any realization *, $T \vdash A^*$.*

The rest of the paper is a proof of this theorem. It has a lot of similarity with proofs given in [3], [4], [11]. Just as in [3] and [4], I define here a Solovay function in terms of regular witnesses rather than provability in finite subtheories (as this is done in [1], [7], [11]). Disregarding this difference, my Solovay function is almost the same as the one given in [11], for both works, unlike [1] or [7], employ finite Veltman models rather than infinite Visser models.

The ($\Longrightarrow$) part can be checked by a routine induction on ILM-proofs, and we are going to prove here only the ($\Longleftarrow$) part.

Suppose $ILM \not\vdash A$. Then, by Theorem 1, there is a finite Veltman model $\langle W, R, \{S_w\}_{w \in W}, \models \rangle$ in which A is not valid. We may assume that $W = \{1, \ldots, l\}$, 1 is the root of the model in the sense that $1Rw$ for all $1 \neq w \in W$, and $1 \not\models A$.

We define a new frame $\langle W', R', \{S'_w\}_{w \in W'} \rangle$:

$W' = W \cup \{0\}$;

$R' = R \cup \{(0, w): \quad w \in W\}$;

$S'_0 = S_1 \cup \{(1, w): \quad w \in W\}$ and for each $w \in W$, $S'_w = S_w$.

Observe that $\langle W', R', \{S'_w\}_{w \in W'}\rangle$ is a finite Veltman frame.

Following the "traditional" way of arithmetical completeness proofs, we are going to embed this frame into T by means of a Solovay [9] style function $g: \quad \omega \to W'$ and sentences $Lim_w \quad (w \in W')$ which assert that w is the limit of g. This function will be defined in such a way that the following basic lemma holds:

Lemma 3

a) *T proves that g has a limit in W', i.e. $T \vdash \bigvee\{Lim_r: \quad r \in W'\}$.*

b) *If $w \neq u$, then $T \vdash \neg(Lim_w \wedge Lim_u)$.*

c) *If $wR'u$, then $T + Lim_w$ proves that $T \nvdash \neg Lim_u$.*

d) *If $w \neq 0$ and not $wR'u$, then $T + Lim_w$ proves that $T \vdash \neg Lim_u$.*

e) *If uS'_wv, then $T + Lim_w$ proves that $T + Lim_v$ is Π_1-conservative over $T + Lim_u$.*

f) *Suppose $wR'u$ and V is a subset of W' such that for no $v \in V$ do we have uS_wv. Then $T + Lim_w$ proves that $T + \bigvee\{Lim_v: \quad v \in V\}$ is not Π_1-conservative over $T + Lim_u$.*

g) *Lim_0 is true.*

To deduce the main thesis from this lemma, we define a substitution * by setting for each propositional letter p,

$$p^* = \bigvee\{Lim_r: \quad r \in W, \ r \models p\}.$$

Lemma 4 *For any $w \in W$ and any ILM-formula B,*

a) *if $w \models B$, then $T + Lim_w \vdash B^*$;*

b) *if $w \not\models B$, then $T + Lim_w \vdash \neg B^*$.*

PROOF by induction on the complexity of B. If B is atomic, then the clause (a) is evident and the clause (b) is also clear in view of 3b. The cases when B is a Boolean combination are straightforward; and since $\Box C$ is ILM-equivalent to $(\neg C) \rhd \bot$, it is enough to consider only the case when $B = C_1 \rhd C_2$.

Assume $w \in W$. Then we can always write wRx and xS_wy instead of $wR'x$ and xS'_wy.

Let $\alpha_i = \{r: \quad wRr, \ r \models C_i\} \quad (i = 1, 2)$.

First we establish that for each $i = 1, 2$,

$$(*) \qquad T + Lim_w \text{ proves that } T \vdash C_i^* \leftrightarrow \bigvee\{Lim_r: \quad r \in \alpha_i\}.$$

Indeed, argue in $T + Lim_w$. Since each $r \in \alpha_i$ forces C_i, we have by the induction hypothesis (clause (a)) that for each such r, $T \vdash Lim_r \rightarrow C_i^*$, whence $T \vdash \bigvee\{Lim_r : \ r \in \alpha_i\} \rightarrow C_i^*$. Next, according to 3a, $T \vdash \bigvee\{Lim_r : \ r \in W'\}$ and, according to 3d, T disproves every Lim_r with *not* wRr; consequently, $T \vdash \bigvee\{Lim_r : \ wRr\}$; at the same time, by the induction hypothesis (clause (b)), C_i^* implies in T the negation of each Lim_r with $r \not\models C_i$. We conclude that $T \vdash C_i^* \rightarrow \bigvee\{Lim_r : \ wRr, \ r \models C_i\}$, i.e. $T \vdash C_i^* \rightarrow \bigvee\{Lim_r : \ r \in \alpha_i\}$. (*) is thus proved. Now continue:

(a) Suppose $w \models C_1 \rhd C_2$. Argue in $T + Lim_w$. By (*), to prove that $T + C_2^*$ is Π_1-conservative over $T + C_1^*$, it is enough to show that $T + \bigvee\{Lim_r : \ r \in \alpha_2\}$ is Π_1-conservative over $T + \bigvee\{Lim_r : \ r \in \alpha_1\}$. Consider an arbitrary $u \in \alpha_1$ (the case with empty α_1 is trivial, for any theory is conservative over $T + \perp$). Since $w \models C_1 \rhd C_2$, there is $v \in \alpha_2$ such that uS_wv. Then, by 3e, $T + Lim_v$ is Π_1-conservative over $T + Lim_u$. Then so is $T + \bigvee\{Lim_r : \ r \in \alpha_2\}$ (which is weaker than $T + Lim_v$). Thus, for each $u \in \alpha_1$, $T + \bigvee\{Lim_r : \ r \in \alpha_2\}$ is Π_1-conservative over $T + Lim_u$. Clearly this implies that $T + \bigvee\{Lim_r : \ r \in \alpha_2\}$ is Π_1-conservative over $T + \bigvee\{Lim_r : \ r \in \alpha_1\}$.

(b) Suppose $w \not\models C_1 \rhd C_2$. Let us then fix an element u of α_1 such that for no $v \in \alpha_2$ do we have uS_wv. Argue in $T + Lim_w$. By 3f, $T + \bigvee\{Lim_r : \ r \in \alpha_2\}$ is not Π_1-conservative over $T + Lim_u$. Then neither is it Π_1-conservative over $T + \bigvee\{Lim_r : \ r \in \alpha_1\}$ (which is weaker than $T + Lim_u$). This means by (*) that $T + C_2^*$ is not Π_1-conservative over $T + C_1^*$. Q.E.D.

Now we can pass to the desired conclusion: since $1 \not\models A$, Lemma 4 gives $T \vdash Lim_1 \rightarrow \neg A^*$, whence $T \not\vdash \neg Lim_1 \Longrightarrow T \not\vdash A^*$. But we have $T \not\vdash \neg Lim_1$ because this fact is derivable in the sound theory T from the true (according to 3g) sentence Lim_0.

Our remaining duty now is to define the function g and prove Lemma 3. The recursion theorem enables us to define this function simultaneously with the sentences Lim_w (for each $w \in W'$), which, as we have mentioned already, assert that w is the limit of g, and formulas $\Delta_{wu}(y)$ (for each pair (w, u) with $wR'u$), which we define by

$$\Delta_{wu}(y) \equiv \exists t > y(g(t) = \bar{u} \wedge \forall z(y \leq z < t \rightarrow g(z) = \bar{w})).$$

Definition 5 (of the function g)

We define $g(0) = 0$.

Assume now $g(y)$ has been defined for every $y \leq x$, and let $g(x) = w$. Then $g(x+1)$ is defined as follows:

1. Suppose $wR'u$, $n \leq x$ and for all z with $n \leq z \leq x$ we have $g(z) = w$. Then, if $\vdash_x Lim_u \rightarrow \neg\Delta_{wu}(\bar{n})$, we define $g(x+1) = u$.

2. Else, suppose $m \leq x$, F is a Π_1!-sentence and the following holds:

a) F has a regular counterwitness which is $\leq x$;

b) $\vdash_m Lim_u \to F$;

c) $wS_{g(m)}u$;

d) m is the least number for which such F and u exist, i.e. there are no m' : $m' < m$, world u' and Π_1!-sentence F' satisfying the conditions (a)–(c) when m', u' and F' stand for m, u and F.

Then we define $g(x+1) = u$.

3. In all the remaining cases $g(x+1) = g(x)$.

It is not hard to see that g is primitive recursive.

Before we start proving Lemma 3, let us agree on some jargon and prove two auxiliary lemmas.

When the transfer from $w = g(x)$ to $u = g(x+1)$ is determined by 5.1, we say that at the moment $x+1$ the function g makes (or we make) an R'*-move* from the world w to the world u. If this transfer is determined by 5.2, then we say that an S'*-transfer* takes place and call the number m from 5.2 the *rank* of this S'-transfer. Sometimes the S'-transfer leads to a new world, but "mostly" it does not, i.e. $(u =)g(x+1) = g(x)(= w)$, and then it is not a move in the proper sense. Those S'-transfers which lead to a new world we call S'*-moves*. As for R'-transfers, they (by irreflexivity of R') always lead to a new world, so we always say "R'-move" instead of "R'-transfer".

In these terms, the formula $\Delta_{wu}(n)$ asserts that beginning from the moment n (but perhaps also before this moment) and until some moment t, we stay at the world w without any motion and then, at the moment t, we move directly to u.

Intuitively, we make an R'-move from w to u, where $wR'u$, in the following situation: since some moment n and up to now we have been staying at the world w, and at the present moment we have reached evidence that $T + Lim_u$ thinks that the first (proper) move which happens after passing the moment n (and thus our next move) cannot lead directly to the world u; then, to spite this belief of $T + Lim_u$, we just move to u.

And the conditions for an S'-transfer from w to u can be described as follows: We are staying at the world w and by the present moment we have reached evidence that $T + Lim_u$ proves a false Π_1!-sentence F. This evidence consists of two components: 1) a regular counterwitness, which indicates that F is false, and 2) the rank m of the transfer, which indicates that $T + Lim_u \vdash F$. Then, as soon as $wS_{g(m)}u$, the next moment we must be at u (move to u, if $u \neq w$, and remain at w, if $u = w$); if there are several possibilities of this transfer, we choose the one with the least rank. Besides, the necessary condition for an S'-transfer is that in the given situation an R'-move is impossible.

Lemma 6 $(T \vdash:)$ *For each natural number m and each $w \in W'$, $T + Lim_w$ proves that no S'-transfer to w can have rank which is less than m.*

PROOF. Indeed, "the rank of an S'-transfer is $< m$" means that $T + Lim_w$ proves a false (i.e. one with a regular counterwitness) Π_1!-sentence F and the code of this proof (i.e. of the T-proof of $Lim_w \to F$) is smaller than m. But the number of all Π_1!-sentences with such short proofs is finite, and as $T + Lim_w$ proves each of them, it also proves that none of these sentences has a regular counterwitness (recall our assumptions about the formula $Regwitness(x, y)$).

Lemma 7 *($T \vdash$:) If $g(x)R'w$, then for all $y \leq x$, $g(y)R'w$.*

PROOF. Suppose $g(x)R'w$ and $y \leq x$. We proceed by induction on $n = x - y$. If $y = x$, we are done. Suppose now $g(y+1)R'w$. If $g(y) = g(y+1)$, we are done. If not, then at the moment $y+1$ the function makes either an R'-move or an S'-move. In the first case we have $g(y)R'g(y+1)$ and, by transitivity of R', $g(y)R'w$; in the second case we have $g(y)S'_v g(y+1)$ for some v, and the desired thesis then follows from the property 5 of Veltman frames.

PROOF OF LEMMA 3. In each case below, except (g), we reason in T.

(a):

First observe that there is z such that for all $z' \geq z$, not $g(z')R'g(z'+1)$.

Indeed, suppose this is not the case. Then, by Lemma 7, for all z there is z' with $g(z)R'g(z')$. This means that there is an infinite (or "sufficiently long") chain $w_1 R' w_2 R' \ldots$, which is impossible because W' is finite and R' is transitive and irreflexive.

So, let us fix this number z. Then we never make an R'-move after the moment z. We claim that S'-moves can also take place at most a finite number of times (whence it follows that g has a limit and this limit is, of course, one of the elements of W').

Indeed, let x be an arbitrary moment after z at which we make an S'-move, and let m be the rank of this move. Taking into account reflexivity of the relations S_w, a little analysis of the condition 5.2 convinces us that the rank of each next S'-move is less than that of the previous one, so S'-moves can take place at most m times after passing x.

(b):

Clearly g cannot have two different limits w and u.

(c):

Assume w is the limit of g and $wR'u$. Let n be such that for all $x \geq n$, $g(x) = w$. We need to show that $T \nvdash \neg Lim_u$. Deny this. Then $T \vdash Lim_u \to \neg\Delta_{wu}(\bar{n})$ and, since every provable formula has arbitrary long proofs, there is $x \geq n$ such that $\vdash_x Lim_u \to \neg\Delta_{wu}(\bar{n})$; but then, according to 5.1, we must have $g(x+1) = u$, which, as $u \neq w$ (by irreflexivity of R'), is a contradiction.

(d):

Assume $w \neq 0$, w is the limit of g and not $wR'u$.

If $u = w$, then (since $w \neq 0$) there is x such that $g(x) = v \neq u$ and $g(x+1) = u$. This means that at the moment $x+1$ we make either an R'-move or an S'-move. In the first case we have $T \vdash Lim_u \to \neg\Delta_{vu}(\bar{n})$ for some n for which, as it is easy to see, the Σ_1!-sentence $\Delta_{vu}(\bar{n})$ is true, whence, by Σ_1!-completeness, $T \vdash \neg Lim_u$. And if an S'-move is the case, then again $T \vdash \neg Lim_u$ because $T + Lim_u$ proves a false (with a $\leq x$ regular counterwitness) Π_1!-sentence.

Suppose now $u \neq w$. Let us fix a number z with $g(z) = w$. Since g is primitive recursive, T proves that $g(z) = w$.

Now argue in $T + Lim_u$: Since u is the limit of g and $g(z) = w \neq u$, there is a number x with $x \geq z$ such that $g(x) \neq u$ and $g(x+1) = u$. Since not $(w =) g(z)R'u$, we have by Lemma 7 that

(*) for each y with $z \leq y \leq x$, not $g(y)R'u$.

In particular, not $g(x)R'u$ and the transfer from $g(x)$ to $g(x+1)(= u)$ can be determined only by 5.2. Then (*) together with the property 3 of Veltman frames and 5.2c, implies that the rank of this S'-move is less than z, which, by Lemma 6, is a contradiction.

Thus, $T + Lim_u$ is inconsistent, i.e. $T \vdash \neg Lim_u$.

(e):

Assume $uS'_w v \neq u$ (the case $v = u$ is trivial). Suppose w is the limit of g, F is a Π_1-sentence and $T \vdash_z Lim_v \to F$. We may suppose that $F \in \Pi_1!$ and that z is sufficiently large, namely, $g(z) = w$. Fix this z. We need to show that $T + Lim_u \vdash F$.

Argue in $T + Lim_u$. Suppose not F. Then there is a regular counterwitness c for F. Let us fix a number $x > z, c$ such that $g(x) = g(x+1) = u$ (as u is the limit of g, such a number exists). Then, according to 5.2, the only reason for $g(x+1) = u \neq v$ can be that we make an S'-transfer from u to u and the rank of this transfer is less than z, which, by Lemma 6, is not the case. Conclusion: F (is true).

(f):

Assume w is the limit of g, $wR'u$, $V \subseteq W'$ and for each $v \in V$, not $uS'_w v$.

Let n be such that for all $z \geq n$, $g(z) = w$. By primitive recursiveness of g, T proves that $g(n) = w$. By 5.1, $T + Lim_u \nvdash \neg\Delta_{wu}(\bar{n})$. So, as $\neg\Delta_{wu}(\bar{n})$ is a Π_1-sentence, in order to prove that $T + \bigvee\{Lim_v : \ v \in V\}$ is not Π_1-conservative over $T + Lim_u$, it is enough to show that for each $v \in V$, $T + Lim_v \vdash \neg\Delta_{wu}(\bar{n})$. Let us fix any $v \in V$. According to our assumption, not $uS'_w v$ and, by reflexivity of S'_w, $u \neq v$.

Argue in $T + Lim_v$. Suppose, for a contradiction, that $\Delta_{wu}(n)$ holds, i.e. there is $t > n$ such that $g(t) = u$ and for all z with $n \leq z < t$, $g(z) = w$. As

v is the limit of g and $v \neq u$, there is $t' > t$ such that $g(t'-1) \neq v$ and at the moment t' we arrive to v to stay there for ever. Let then $x_0 < \ldots < x_k$ be all the moments in the interval $[t, t']$ at which R'- or S'-moves take place, and let $u_0 = g(x_0), \ldots, u_k = g(x_k)$. Thus $t = x_0,\ \ t' = x_k,\ \ u = u_0,\ \ v = u_k$ and $u_0, \ldots, u_k$ is the route of g after departing from w (at the moment t).

Let now j be the least number among $1, \ldots, k$ such that for all $j \leq i \leq k$, not $u_0 R' u_i$. Note that such a j does exist because at least $j = k$ satisfies this condition (otherwise, if $(u =) u_0 R' u_k (= v)$, the property 4 of Veltman frames would imply $u S'_w v$).

Note also that for each i with $j \leq i \leq k$, the move to u_i cannot be an R'-move. Indeed, otherwise we must have $u_{i-1} R' u_i$, whence, by Lemma 7, $u_0 R' u_i$, which is impossible for $i \geq j$.

Thus, beginning from the moment x_j (inclusive), each move is an S'-move. Moreover: for each i with $j \leq i \leq k$, the rank of the S'-move to u_i is less than x_0. For otherwise the property 3 of Veltman frames together with Lemma 7 would give by 5.2c that $u_0 R' u_i$. On the other hand, since consecutive S'-moves decrease the rank (as we noted in the proof of (a) above) and since the rank of the S'-move to u_k cannot be less than n (Lemma 6), we conclude: for each i with $j \leq i \leq k$, the rank of the S'-move to u_i is in the interval $[n, x_0 - 1]$. But the value of g in this interval is w, and by 5.2c this means that $u_{j-1} S'_w u_j S'_w \ldots S'_w u_k$. At the same time, we have either $u_0 = u_{j-1}$ or $u_0 R' u_{j-1}$. In both cases we then have $u_0 S'_w u_{j-1}$ (in the first case by reflexivity of S'_w and in the second case by the property 4 of Veltman frames), whence, by transitivity of S'_w, $\ u_0 S'_w u_k$, i.e. $u S'_w v$, which is a contradiction.

Conclusion: $T + Lim_v \vdash \neg \Delta_{wu}(\bar{n})$.

(g):

By 3a, as T is sound, one of the $Lim_w\ \ (w \in W')$ is true. Since for no w do we have $wR'w$, 3d means that each Lim_w, except Lim_0, implies in T its own T-disprovability and therefore is false. Consequently, Lim_0 is true. Q.E.D.

This, in turn, completes the proof of Theorem 2.

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