Open problems in computability logic

Giorgi Japaridze

January 6, 2012

This list of open problems is incomplete, and will be continuously improved and updated. The degrees of
difficulty or importance may vary significantly from problem to problem. I however believe that a solution
of any of these problems, if written in a satisfactory manner, would make a publication in a decent journal.

1. Remark 7 of [6] claims that the cirquent calculus system CL5 without duplication has polynomial
size proofs (every provable formula $F$ has a proof whose size is polynomial in the size of $F$). Verify
this claim. What kind of a polynomial do we have here? Does such a system have any reasonable
semantics?

2. Consider the system CCC from [6] but without the Weakening rule. Does it have an interesting
semantics? (Hint: think of Relevance Logic).

3. The same question as above for CCC with both Weakening and Contraction deleted (such a system is
still stronger than linear logic as, for instance, it proves Blass’s principle).

4. Verify Conjecture 10.2 of [14], at least for uniform (strong) validity (this case is much simpler than the
case of just (weak) validity, yet it is at least equally important).

5. Strengthen the results of [10] (the soundness and completeness of the implicative fragment of intuition-
istic logic) by adding \( \bot \) to the vocabulary. My expectation is that the \( (\circlearrowleft, \bot) \)-fragment of intuitionistic
logic remains sound and complete with \( \circlearrowleft \) understood as either \( \circlearrowleft \)– or \( \Rightarrow \)–.

6. Paper [12] proved the soundness and completeness of the full propositional intuitionistic logic. However,
the intuitionistic absurd in it was understood as \( \$ \) rather than (the more natural) \( \bot \). If we change \( \$ \) to \( \bot \), we get a superintuitionistic logic. Is that logic decidable or recursively enumerable? If so, try to
 axiomatize it. Looking at the corresponding Kripke semantics would also be very interesting.

7. Consider the language of the logic CL1 from [4] with the additional operator \( \circlearrowleft | \) (and its dual \( \circlearrowright | \)). I
expect that the set of valid (or uniformly valid) formulas of this language is decidable. Verify this, and
try to construct a corresponding axiomatization.

8. Do the same as in the previous problem, but for \( \land | \) instead of \( \circlearrowleft | \).

9. Is the logic CL2 from [5] PSPACE-complete?

10. Extend the cirquent calculus system CL5 from [6] so as to get a sound and complete (and proof-
theoretically reasonable) system for the \( (\neg, \land, \lor, \top, \bot) \)-fragment of computability logic.

11. Do the same as above for the \( (\neg, \land, \lor, \Delta, \nabla) \)-fragment (see [14] for \( \Delta, \nabla \)).

12. Do the same for the system CL8 from [13]. A semantic setup can be found in [20].

13. Does the above system CL8 allow cut elimination without an exponential growth of proof sizes?

14. Extend the language of the system CL12 from [19] through including general letters (CL12 only has
elementary letters). Try to axiomatize the set of (weakly or strongly) valid principles in this language.
Hint: Combine the approaches of [19] and [9].
15. In proving the completeness of CL12, [19] appeals to what it calls non-ideal universes. Restricting attention to ideal universes yields a stronger logic. For instance, the latter validates $x \equiv y \cup x \neq y$, which is not provable in CL12. Try to axiomatize such a “stronger logic”.

16. Is it true that a formula in the signature $(\neg, \wedge, \diamond)$ is valid (“weakly valid”) iff it is uniformly valid (“strongly valid”)?

17. The same question as above for various other, recurrence-involving signatures, such as $(\neg, \wedge, \lambda)$, $(\neg, \wedge, \diamond, \sqcap)$, signatures with quantifiers, etc. (For essentially all recurrence-free signatures, the question has been shown to have a positive answer).

18. Is the $(\neg, \wedge, \diamond)$-fragment of (the set of valid formulas of) computability logic decidable or recursively enumerable? If yes, try to find an axiomatization.

19. The same question as above for various other, recurrence-involving fragments, such as $(\neg, \wedge, \lambda)$, $(\neg, \wedge, \diamond, \sqcap)$, etc.

20. Prove that the set of static games is closed under toggling-branching recurrence (introduced in [18]).
   Hint: Look at a similar proof for branching recurrence given in [3].

21. Verify Claim 4.5 of [18].

22. Extend the language of the logic CL13 from [18] by adding the quantifiers $\forall, \exists, \sqcap, \sqcup$, and try to axiomatize the set of (uniformly) valid formulas in the resulting language. Hint: Combine the approaches of [18] and [9].

23. Is the theory CLA1 from [17] a conservative extension of PA?

24. Is there a sentence provable in the theory CLA1 from [17] but not in the theory CLA8 from [24]?

25. The theories CLA4, CLA5, CLA6, CLA7 (from [22, 23]) are sound and extensionally complete with respect to polynomial time computability, polynomial space computability, elementary recursive time/space computability and primitive recursive time/space computability, respectively. In the same sense, the theory CLA1 from [17] is sound and complete with respect to $X$ time/space computability. What class of functions is this $X$? (I am only aware that $X$ is properly bigger than the class of primitive recursive functions).

26. Try to construct a theory sound and extensionally complete with respect to logarithmic space computability (in the same sense as, say, the above-mentioned theory CLA5 is sound and complete with respect to polynomial space computability). Do the same for the various classes of the polynomial hierarchy.

27. So far all completeness proofs for various systems with respect to validity (as opposed to uniform validity) have appealed to imperfect (as opposed to perfect) interpretations — interpretations that do not respect the arities of the elementary or general letters. Restricting interpretations to perfect ones creates certain “anomalies” — namely, not every valid formula is also uniformly valid — if the language under consideration contains elementary atoms. The same, however, does not appear to be the case if we consider general-base languages, i.e. languages that only have general atoms. Try to prove the completeness of the general-base fragment of the system CL2 from [5] with respect to perfect interpretations. Hint: Helpful ideas can be found in [1].

28. Do the same for the system CL4 from [9].

29. In the same spirit, prove the version of Theorem 10.1 of [20] in which $C$ is a cirquent that has no elementary ports, and which talks about weak validity instead of strong validity, with “weak validity” understood as computability under every perfect interpretation.

30. Section 7 of [20] conjectures that the so called ranked IF logic is properly more expressive than extended IF logic. Verify (or refute) this conjecture.
31. Section 10 of [16] outlines potential applications of computability logic in knowledge base systems. Further elaborate that line, and make first concrete steps towards materializing it.

32. Develop reasonable theorem-provers for the system CL12 from [19].

33. Develop reasonable theorem-provers for the theory CLA4 from [22].

34. Write a good survey-style paper on computability logic versus linear logic. The relationship between the two, and the relative advantages/disadvantages are to be better understood.

35. Write a good survey-style paper on computability logic versus the other game-semantical approaches.

36. Is the logic CL15 from [26, 27] decidable?

References


