On cliques in edge-regular graphs

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Modern Trends in Algebraic Graph Theory, Villanova, 2014
The setup

All graphs in this talk are finite, undirected, and have no loops and no multiple edges.
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**Definition**

A graph $\Gamma$ is *edge-regular* with *parameters* $(v, k, \lambda)$ if $\Gamma$ has exactly $v$ vertices, is regular of valency $k$, and every pair of adjacent vertices have exactly $\lambda$ common neighbours.
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A clique in a graph $\Gamma$ is a set of pairwise adjacent vertices, an $s$-clique is a clique of size $s$, and a maximum clique of $\Gamma$ is a clique of the largest size in $\Gamma$. The size of a maximum clique in $\Gamma$, its clique number, is denoted by $\omega(\Gamma)$. 
Definition

A *regular clique*, or more specifically, an *$m$-regular clique* in a graph $\Gamma$ is a non-empty clique $S$ such that every vertex of $\Gamma$ not in $S$ is adjacent to exactly $m$ vertices of $S$, for some constant $m > 0$. 

A *quasiregular clique*, or more specifically, an *$m$-quasiregular clique* in a graph $\Gamma$ is a clique $S$ of size at least 2, such that every vertex of $\Gamma$ not in $S$ is adjacent to exactly $m$ or $m+1$ vertices of $S$, for some constant $m \geq 0$. 

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Cliques in ERGs

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I am also interested in properties of quasiregular cliques in edge-regular graphs.
The clique adjacency polynomial

This is our main tool.

**Definition**

The *clique adjacency polynomial* of an edge-regular graph $\Gamma$ with parameters $(v, k, \lambda)$ is $C_\Gamma(x, y) = C_{v,k,\lambda}(x, y) :=$

$$x(x + 1)(v - y) - 2xy(k - y + 1) + y(y - 1)(\lambda - y + 2).$$
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This polynomial is a special case of the “block intersection polynomials” introduced by Cameron and S. (2007) and further studied by S. (2010). The theory of block intersection polynomials can be used to prove the following:
Theorem

Let $\Gamma$ be an edge-regular graph with parameters $(v, k, \lambda)$, let $C(x, y) = C_{v,k,\lambda}(x, y)$, and suppose $\Gamma$ has an $s$-clique $S$, with $s \geq 2$. Then:

1. $C(x, s) = \sum_{i=0}^{s} (i - x)(i - x - 1)n_i$, where $n_i$ is the number of vertices of $\Gamma$ not in $S$ adjacent to exactly $i$ vertices in $S$;

2. $C(m, s) \geq 0$ for every integer $m$;

3. if $m$ is a non-negative integer then $C(m, s) = 0$ if and only if $S$ is $m$-quasiregular, in which case the number of vertices outside $S$ adjacent to exactly $m$ vertices in $S$ is $C(m + 1, s)/2$;

4. if $m$ is a positive integer then $C(m - 1, s) = C(m, s) = 0$ if and only if $S$ is $m$-regular.
We can apply the clique adjacency polynomial to prove the following theorem, which generalises a result of Neumaier (1981) on regular cliques in edge-regular graphs.

**Theorem**

Suppose $\Gamma$ is an edge-regular graph, not complete multipartite, which has an $m$-quasiregular $s$-clique. Then for all edge-regular graphs $\Delta$ with the same parameters $(v, k, \lambda)$ as $\Gamma$:

1. $\omega(\Delta) \leq s$, so in particular, $\omega(\Gamma) = s$;

2. all quasiregular cliques in $\Delta$ are $m$-quasiregular cliques;

3. the quasiregular cliques in $\Delta$ are precisely the cliques of size $s$ (although $\Delta$ may have no cliques of size $s$).
A clique adjacency polynomial can be used to determine an upper bound on the clique number of an edge-regular graph $\Gamma$ with given parameters $(v, k, \lambda)$, as follows.

Let $C(x, y) = C_{v, k, \lambda}(x, y)$, and let $b = b_{v, k, \lambda}$ be the least positive integer such that $C(m, b+1) < 0$ for some integer $m$. Then $\omega(\Gamma) \leq b$.

Such a $b$ always exists, and is easy to calculate.

I know of no case of a strongly regular graph with parameters $(v, k, \lambda, \mu)$ where the bound $b = b_{v, k, \lambda}$ is worse than the Delsarte-Hoffman bound, and some cases where the bound $b$ is strictly better.
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Bounding the clique number of an edge-regular graph

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Example

The parameters with smallest $v$ for which the existence of a strongly regular graph is unknown are

$$(v, k, \lambda, \mu) = (65, 32, 15, 16).$$

A strongly regular graph with these parameters would have least eigenvalue $(-1 - \sqrt{65})/2$, and the Delsarte-Hoffman bound would be

$$8 = \left\lfloor 1 + 64/(1 + \sqrt{65}) \right\rfloor.$$

However, we calculate that $b_{65, 32, 15} = 7$ (in particular, $C_{65,32,15}(3, 8) = -12$), and so any edge-regular graph $\Delta$ with parameters $(65, 32, 15)$ has $\omega(\Delta) \leq 7$. 
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However, we calculate that \( b_{65,32,15} = 7 \) (in particular, \( C_{65,32,15}(3, 8) = -12 \)), and so any edge-regular graph \( \Delta \) with parameters \((65, 32, 15)\) has \( \omega(\Delta) \leq 7 \).

Perhaps it would be fruitful to search for a strongly regular graph with parameters \((65, 32, 15, 16)\) and containing a clique of size 7.
And finally

Some references:


