Open problems in computability logic

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This list of open problems is incomplete, and will be continuously improved and updated. The degrees of difficulty or importance may vary significantly from problem to problem. I however believe that a solution of any of these problems, if written in a satisfactory manner, would make a publication in a decent journal.

1. Remark 7 of [6] claims that the cirquent calculus system **CL5** without duplication has polynomial size proofs (every provable formula F has a proof whose size is polynomial in the size of F). Verify this claim. What kind of a polynomial do we have here? Does such a system have any reasonable semantics?

2. Consider the system **CCC** from [6] but without the Weakening rule. Does it have an interesting semantics? (*Hint*: think of Relevance Logic).

3. The same question as above for **CCC** with *both* Weakening and Contraction deleted (such a system is still stronger than linear logic as, for instance, it proves Blass's principle).

4. Verify Conjecture 10.2 of [14], at least for uniform (strong) validity (this case is much simpler than the case of just (weak) validity, yet it is at least equally important).

5. Strengthen the results of [10] (the soundness and completeness of the implicative fragment of intuitionistic logic) by adding $\bot$ to the vocabulary. My expectation is that the $(\supset, \bot)$-fragment of intuitionistic logic remains sound and complete with $\supset$ understood as either ∘– or ≻–.

6. Paper [12] proved the soundness and completeness of the full propositional intuitionistic logic. However, the intuitionistic absurd in it was understood as \$ rather than (the more natural) $\bot$. If we change \$ to $\bot$, we get a superintuitionistic logic. Is that logic decidable or recursively enumerable? If so, try to axiomatize it. Looking at the corresponding Kripke semantics would also be very interesting.

7. Consider the language of the logic **CL1** from [4] with the additional operator ⫰ (and its dual ⫯). I expect that the set of valid (or uniformly valid) formulas of this language is decidable. Verify this, and try to construct a corresponding axiomatization.

8. Do the same as in the previous problem, but for ⅄ instead of ⫰.

9. Is the logic **CL2** from [5] PSPACE-complete?

10. Extend the cirquent calculus system CL5 from [6] so as to get a sound and complete (and proof-theoretically reasonable) system for the $(\neg, \wedge, \vee, \sqcap, \sqcup)$-fragment of computability logic.

11. Do the same as above for the $(\neg, \wedge, \vee, \triangle, \triangledown)$-fragment (see [14] for $\triangle, \triangledown$).

12. Do the same for the system **CL8** from [13]. A semantic setup can be found in [20].

13. Does the above system **CL8** allow cut elimination without an exponential growth of proof sizes?

14. Extend the language of the system **CL12** from [19] through including general letters (**CL12** only has elementary letters). Try to axiomatize the set of (weakly or strongly) valid principles in this language. *Hint*: Combine the approaches of [19] and [9].

15. In proving the completeness of **CL12**, [19] appeals to what it calls *non-ideal universes*. Restricting attention to ideal universes yields a stronger logic. For instance, the latter validates $x = y \sqcup x \neq y$, which is not provable in **CL12**. Try to axiomatize such a "stronger logic".

16. Is it true that a formula in the signature (¬, ∧, ⫰) is valid ("weakly valid") iff it is uniformly valid ("strongly valid")?

17. The same question as above for various other, recurrence-involving signatures, such as (¬, ∧, ⅄), (¬, ∧, ⫰, ⊓), signatures with quantifiers, etc. (For essentially all recurrence-free signatures, the question has been shown to have a positive answer).

18. Is the (¬, ∧, ⫰)-fragment of (the set of valid formulas of) computability logic decidable or recursively enumerable? If yes, try to find an axiomatization.

19. The same question as above for various other, recurrence-involving fragments, such as (¬, ∧, ⅄), (¬, ∧, ⫰, ⊓), etc.

20. Prove that the set of static games is closed under toggling-branching recurrence (introduced in [18]). *Hint*: Look at a similar proof for branching recurrence given in [3].

21. Verify Claim 4.5 of [18].

22. Extend the language of the logic **CL13** from [18] by adding the quantifiers $\forall, \exists, \sqcap, \sqcup$, and try to axiomatize the set of (uniformly) valid formulas in the resulting language. *Hint*: Combine the approaches of [18] and [9].

23. Is the theory **CLA1** from [17] a conservative extension of **PA**?

24. Is there a sentence provable in the theory **CLA1** from [17] but not in the theory **CLA8** from [24]?

25. The theories **CLA4**, **CLA5**, **CLA6**, **CLA7** (from [22, 23]) are sound and extensionally complete with respect to polynomial time computability, polynomial space computability, elementary recursive time/space computability and primitive recursive time/space computability, respectively. In the same sense, the theory **CLA1** from [17] is sound and complete with respect to X time/space computability. What class of functions is this X? (I am only aware that X is properly bigger than the class of primitive recursive functions).

26. Try to construct a theory sound and extensionally complete with respect to logarithmic space computability (in the same sense as, say, the above-mentioned theory **CLA5** is sound and complete with respect to polynomial space computability). Do the same for the various classes of the polynomial hierarchy.

27. So far all completeness proofs for various systems with respect to validity (as opposed to uniform validity) have appealed to imperfect (as opposed to perfect) interpretations — interpretations that do not respect the arities of the elementary or general letters. Restricting interpretations to perfect ones creates certain "anomalies" — namely, not every valid formula is also uniformly valid — if the language under consideration contains elementary atoms. The same, however, does not appear to be the case if we consider general-base languages, i.e. languages that only have general atoms. Try to prove the completeness of the general-base fragment of the system **CL2** from [5] with respect to perfect interpretations. *Hint*: Helpful ideas can be found in [1].

28. Do the same for the system **CL4** from [9].

29. In the same spirit, prove the version of Theorem 10.1 of [20] in which C is a cirquent that has no elementary ports, and which talks about weak validity instead of strong validity, with "weak validity" understood as computability under every perfect interpretation.

30. Section 7 of [20] conjectures that the so called ranked IF logic is properly more expressive than extended IF logic. Verify (or refute) this conjecture.

31. Section 10 of [16] outlines potential applications of computability logic in knowledge base systems. Further elaborate that line, and make first concrete steps towards materializing it.

32. Develop reasonable theorem-provers for the system **CL12** from [19].

33. Develop reasonable theorem-provers for the theory **CLA4** from [22].

34. Write a good survey-style paper on computability logic versus linear logic. The relationship between the two, and the relative advantages/disadvantages are to be better understood.

35. Write a good survey-style paper on computability logic versus the other game-semantical approaches.

36. Is the logic **CL15** from [26, 27] decidable?

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