Name: ANSWER KEY

Department of Computing Sciences
CSC 1300 – Final Exam
December 14, 2016 Fall 2016

Show your work carefully. Just writing an answer will not do. Show any and all assumptions; show the steps you took, and show clearly how you came to your answer. (Most of these problems are slightly modified from those submitted by all of you; thank you for providing so many great problems.)

The following items are provided in case you need them.

How many ways ... ... n labeled boxes?	at most one per box	any number per box	exactly one per box
k labeled (ordered) balls	A: $\binom{n}{k}k! = n(n-1)\ldots(n-k+1)$	E, F: (k_j balls unordered within box) $\frac{k!}{k_1!k_2!\ldots k_n!} = \binom{k}{k_1}$ $\binom{k-k_1}{k_2}\binom{k-k_1-k_2}{k_3}$ $\ldots\binom{k_n+k_{n-1}}{k_{n-1}}$	———
k unlabeled (unordered) balls	B: $\binom{n}{k}$	D, D': $\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$ and $\binom{k-1}{n-1} = \binom{k-1}{k-n}$	———
unlimited balls, k different labels (order matters)	———	———	C: k^n

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Name: _____

1. [___/10] (modified Chapter 2 Homework problem) Let A = {1,2,3}, B = {2,3,4,5}
Find the following sets:

a.) A x (B|A) $B \setminus A = \{4,5\}$

so $A \times (B \setminus A) = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

b.) $\mathcal{P}(A) =$ (the "power set" of A)

$\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

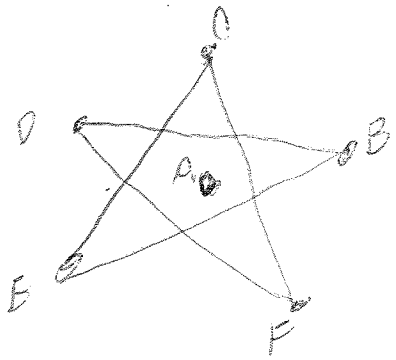
2. [___/10] Given this truth table, write the Conjunctive Normal Form (CNF) for the formula g(A,B,C).

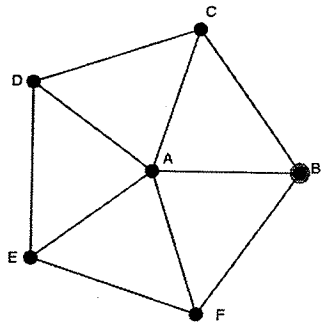
A	B	C	g(A,B,C)
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

$(\neg A \vee \neg B \vee C) \wedge (\neg A \vee B \vee C) \wedge (A \vee \neg B \vee C)$

Name: _____

3. [___/10] Draw the complement of this W_6 graph, then give the adjacency matrix of the complement.





	A	B	C	D	E	F
A	0	0	0	0	0	0
B	0	0	0	1	1	0
C	0	0	0	0	1	1
D	0	1	0	0	0	1
E	0	1	1	0	0	0
F	0	0	1	1	0	0

Name: _____

4. [___/10] a.) Given the set A = { 10, 15, 16, 1, 30, 2, 9, 4}, is the set {B,C} a partition of A, where B = {1, 2, 4, 10} and C = {15, 16, 9}? If not, why not?

30 is not in $B \cup C$ so not a partition,

b.) Let B = {1, 4, 10}, C = {15, 16, 9} and D = {2, 30}. Is the set {B, C, D} a partition of A? If not, why not?

Since $B \cup C \cup D = A$ and $\begin{cases} B \cap C = \emptyset \\ B \cap D = \emptyset \\ C \cap D = \emptyset \end{cases}$

this is a partition of A.

5. [___/10] a) True or false? (Explain your answer): $2 \equiv 10 \pmod{12}$

$2 - 10 = -8$ is not a multiple of 12

b) True or false? (Explain your answer): $2 \equiv -10 \pmod{12}$

$2 - (-10) = 12 = 12 \cdot 1$ is a multiple of 12.

c.) Given $x \equiv y \pmod{n}$ and $u \equiv v \pmod{n}$, prove $xu \equiv yv \pmod{n}$

Since $x - y = n \cdot a$, $x = y + na$

$u - v = n \cdot b$, $u = v + nb$

so $xu - yv = (y + na)(v + nb) - yv$

$= yv + nav + ynb + n^2ab - yv$

$= n(av + yb + nab) = nM$ multiple of N

so $xu \equiv yv \pmod{n}$.

Name: _____

6. [___/20] Give the full proof by induction that for every natural number n,

$$\sum_{j=1}^{n} 2j = n(n+1)$$

Base Case $n=1$ $\quad 2\cdot 1 = 1\cdot(1+1)$ ✓

Inductive Hypothesis. If $\sum_{j=1}^{n} 2j = n(n+1)$

Inductive Consequent then $\sum_{j=1}^{n+1} 2j = (n+1)(n+2)$.

Proof

$$\sum_{j=1}^{n+1} 2j = \left(\sum_{j=1}^{n} 2j\right) + 2(n+1)$$ By the inductive hypothesis

$$= n(n+1) + 2(n+1)$$ Factor out $n+1$

$$= (n+2)(n+1)$$

$$= (n+1)(n+2)$$

Conclusion $\sum_{j=1}^{n} 2j = n(n+1)$ for all n.

Name: _____

7. [___/10] Encrypt the message "Oranges are great" using the Vigenere cipher and key word "lemon".

O	R	A	N	G	E	S	A	R	E	G	R	E	A	T
14	17	0	13	6	4	18	0	17	4	6	17	4	0	19
11	4	12	14	13	11	4	12	14	13	11	4	12	14	13
25	21	12	27	19	15	22	12	31	17	17	21	16	14	32
			1					5						6
Z	V	M	B	T	P	W	M	F	R	R	V	Q	O	G

(LEMON)

← answer

8. [____ / 20] Write out the first six rows of Pascal's triangle, then use this to help find the term in the expansion of $(x^2 + 2y^3)^5$ in which y^6 appears.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$(a+b)^5 = 1\cdot a^5 + 5\cdot a^4 b + 10\cdot a^3 b^2 + 10 a^2 b^3 + 5ab^4 + 1\cdot b^5$

$a = x^2$

$b = 2y^3 \rightarrow b^2 = 4y^6$ Any y^6 factor.

So need $10a^3b^2 = 10(x^2)^3(2y^3)^2 = 40x^6y^6$

Name: _____

9. [___/10] How many anagrams are there (including those that aren't dictionary words) of 'cardinality'? Please explain how you obtained this number.

11 letters, 2 repeated i, 2 repeated a

So $\frac{11!}{2!2!}$ OK $\binom{11}{2}\binom{9}{2}7!$ OR $\binom{11}{7}\cdot 7!\binom{4}{2}2!\cdot 2!$

10. [___/20] a.) How many ways are there to give three different catnip toys to five cats, such that no cat gets more than one toy?

$5\cdot 4\cdot 3 = 60$ OR $\binom{5}{3}\cdot 3!$ OR $\frac{5!}{2!}$

b.) How many ways are there to give three identical catnip toys to five cats, allowing a cat to get more than one toy?

stars and bars $\binom{5+3-1}{3} = \binom{7}{3} = 35$

OR

one cat 3 $\rightarrow \binom{5}{1} = 5$

one cat 2, one 1 $\rightarrow (5)(4) = 20$

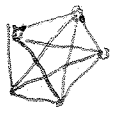
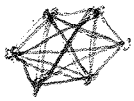
3 cats each 1 $\rightarrow \binom{5}{3} = 10$

mutually exclusive so add

$5+20+10 = 35$

Name: _____

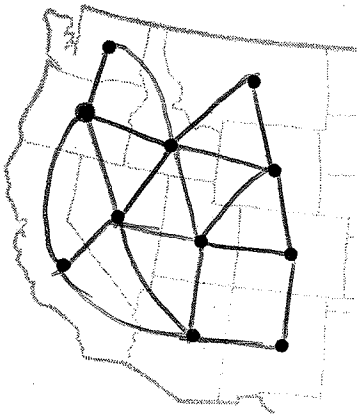
11. [___/10] List the chromatic index for K_3, K_4, K_5, and K_6. What is the general theorem for the chromatic index of K_n for any positive integer n?

K_3  3 K_4  3 K_5  5 K_6  5

K_n if n is odd $\chi'(K_n) = n$

if n is even, $\chi'(K_n) = n-1$

12. [___/20] We will use graph theory to discuss how many colors are required to color a map of the eleven states west of Texas so that no two adjacent regions are given the same color. First, explicitly create a graph with eleven vertices that has an edge whenever two states share a common border. Determine both the chromatic number χ and the chromatic index χ' of this graph. Does the chromatic number or the chromatic index tell you how many colors to use for the map?



Because of the W_6 with Nevada in the center, we need at least four colors, and since it is planar, at most 4, hence $\chi = 4$.

Since the highest degree vertex (Idaho) is 6, we need at least 6 edge colors, and it is easy to see 5 works, so $\chi' = 6$.

For coloring the map, we want $\chi = 4$ colors. (chromatic number)

Name: _____

13. [____ / 20] Solve one of the following recurrence relations using the method of characteristic equations or kth differences. Show all your work carefully.

$a_n = 5a_{n-1} - 4a_{n-2}$, with $a_0=0$, $a_1=1$.

$a_n = a_{n-1} - 4n$, with $a_0=0$, $a_1=1$.

$a_n = 5\,a_{n-1} - 4n$, with $a_0=0$, $a_1=1$.

$a_n = 5a_{n-1} - 4a_{n-2}$ characteristic equation Guess $a_n = r^n$

$r^n = 5r^{n-1} - 4r^{n-2} \longrightarrow r^2 - 5r + 4 = 0$

$(r-1)(r-4) = 0$ so $r = 1, 4$

General solution $a_n = A \cdot 1^n + B \cdot 4^n$

Initial Conditions $a_0 = A \cdot 1^0 + B \cdot 4^0 = A + B = 0$

$a_1 = A \cdot 1^1 + B \cdot 4^1 = A + 4B = 1$

Subtract to get $3B = 1 \longrightarrow B = 1/3$, $A = -1/3$

Answer $a_n = \frac{1}{3} \cdot 4^n - \frac{1}{3}$

OR $a_n = a_{n-1} - 4n$ k^{TH} difference

Guess $a_n = An^2 + Bn + C$ (or assume we know $A=-2$ from calculus)

$a_0 = 0$ so $a_1 = a_0 - 4(1) = -4$

$a_2 = a_1 - 4(2) = -12$

Then $a_0 = A \cdot 0^2 + B \cdot 0 + C = 0 \longrightarrow C = 0$

$a_1 = A \cdot 1^2 + B \cdot 1 + C = -4 \longrightarrow A + B = -4$

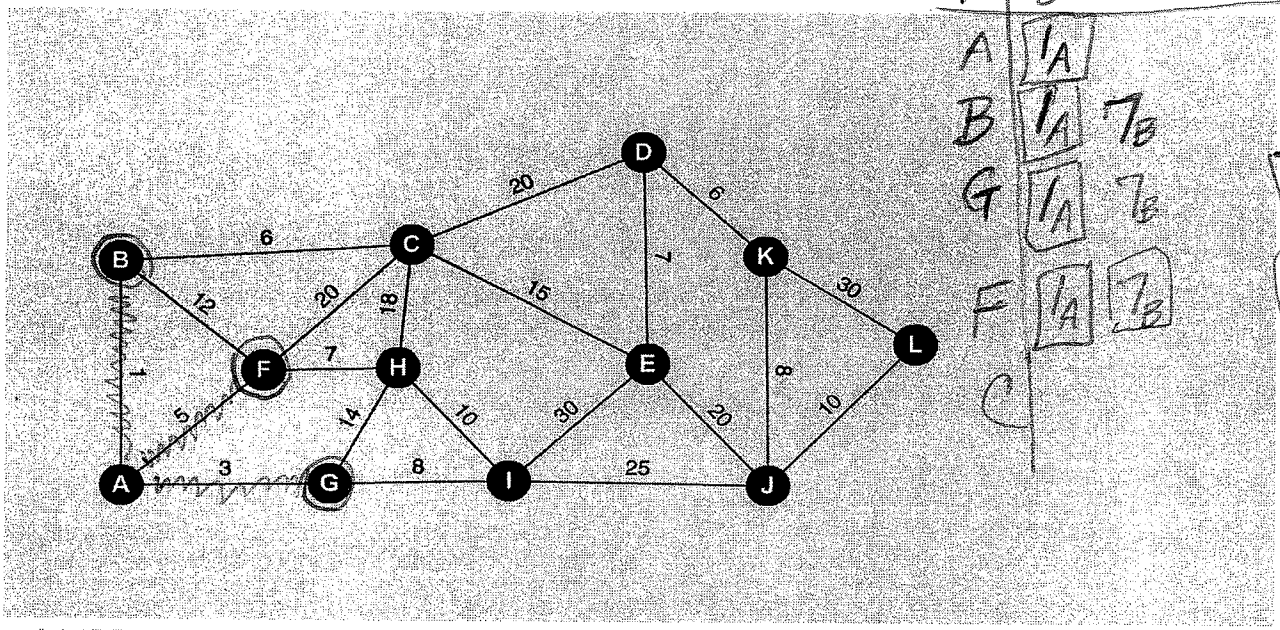
$a_2 = A \cdot 2^2 + B(2) + C = -12 \longrightarrow 4A + 2B = -12$

$A = -2$, $B = -2$

Answer $a_n = -2n^2 - 2n$

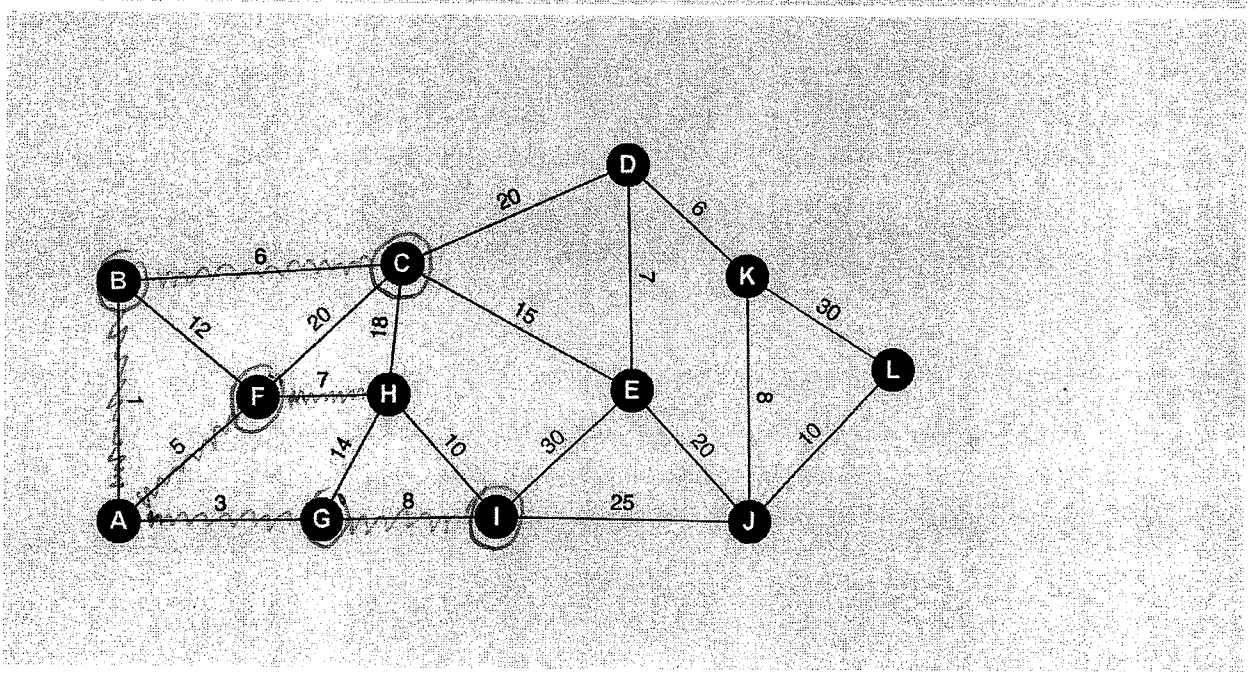
Name: _____

14. [____/20] [Dijkstra algorithm] Given this graph, use Dijkstra's algorithm to find the shortest path from node A to all other nodes. Show your work clearly. Just giving the answer will not be accepted. CLEARLY SHOW EACH STEP OF THE DIJKSTRA ALGORITHM IN ORDER. In case you are accustomed to using extra copies of the graph, a few extra are included. You will not need to use all of these copies.
If you do not wish to show the steps on the graphs, exhibit your trace in a consistent way that *shows the chronology of when each vertex is 'tagged.'*

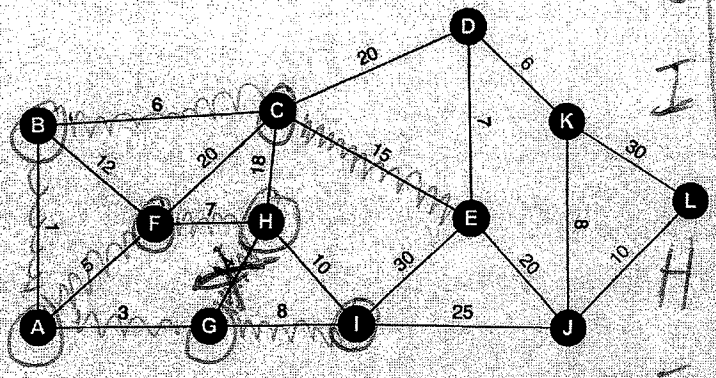


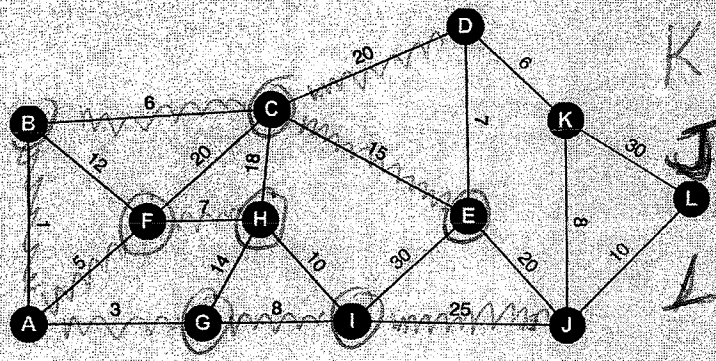
A | B | C | D | E | F | G | H I
A | 1A
B | 1A | 7B
G | 1A | 7B
F | 1A | 7B
C

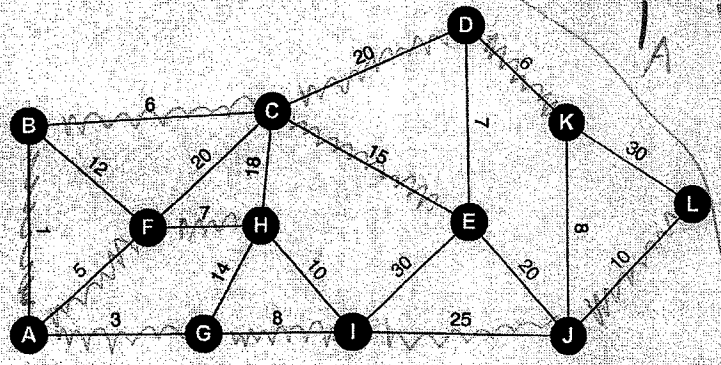
5A 3A
5A 3A
5A 3A 12F 11G
5A 3A 12F 11G



Name: _____







A B C D E F G H I J K L

BGF... C 1_A 7_B 27_C 22_C 5_A 3_A 12_F 11_G

I 1_A 7_B 27_C 22_C 5_A 3_A 12_F 11_G 36_I

H 1_A 7_B 27_C 22_C 5_A 3_A 12_F 11_G 36_I

E 1_A 7_B 27_C 22_C 5_A 3_A 12_F 11_G 36_I ∞ ∞

" " " 36_I 33_D ∞

D " " " 36_I 33_D ~~63_K~~

K " " " " " 46_J

J " " " "

L " " " "

Anjuwar

B	C	D	E	F	G	H
1_A	7_B	27_C	22_C	5_A	3_A	12_F

I	J	K	L
11_G	36_I	33_D	46_J