Name: ______________________________

VILLANOVA UNIVERSITY
Department of Computing Sciences
CSC 1300-002 March 8, 2016
Exam 2 of 3

Show your work carefully. Just writing an answer will not do. Show any assumptions; show the steps you took; and show how you came to your answer. And be sure to write legibly!

You may have use of one index card with whatever information you chose to put on it. No calculators or phones or any other information sources are permitted.

1. (5 pts) Draw a *simple* graph with degree sequence (1,1,2,2,4).

2. (5 pts) Write as a summation (sigma): $x/2 + x^2/6 + x^3/10 + x^4/14 + x^5/18$.

3. (5 pts) Evaluate the following summation. Be sure to <u>*write out all the terms first*</u>. You should end up with a single fraction.

$$\sum_{i=0}^{5} (m^2 - 5m + 6)\,/(m + 1)$$

Name: ____

4. (10 pts) Use induction to show that the following equality holds for all M>=0. Remember to write out each part of the induction proof carefully.

$$\sum_{n=0}^{M} 3\,(n-2)\,(n+1) = (M+2)(M+1)(M-3)$$

Name: ______________________________________

5. For each one, state whether it is an equivalence relation or not. If it is not an equivalence relation, give all properties that fail (and tell why they fail):

   a. (5 pts) On the set of all persons born in the USA, let u~v if and only if u and v were born within one hundred kilometers of each other.

   b. (5 pts) On the set of all persons born in the USA, let u~v if and only if u and v were born in the same state.

   c. (5 pts) On the set {a, b, c, d, e}, let u~v if and only if (u, v) is in the following set
   { (a,a), (b, a), (c,c), (a, c), (b, c), (c, b), (b, b), (c, a), (a, b), (d,d) }

6. (10 pts) Use the definition of congruence modulo N to prove that if $j \equiv k \pmod{N}$ then $j^2 \equiv k^2 \pmod{N}$.

Name: ______________________________________

7. (10 pts) Use the ***standard*** (i.e., easy) Vigenère algorithm to decode the following message. You will find it helpful to have the following table.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

Keyword: CATS

Coded message: (You may choose not to use all the available lines. Do show your work clearly.)

P	E	O	W	T	S	T	Q	P	E	O	W	T

8. Consider the problem: In how many ways could you plan a pizza party, if you want exactly 5 different pizzas from a list of 7 pizza types, and you want exactly 3 types of sodas from a list of 5 soda types?

(a.) (4 pts) Express the solution in choice notation: ____________________________

(b.) (4 pts) Show (at least) the relevant rows of Pascal's triangle:

(c.) (2 pts) Indicate number of ways the party can be planned: ______________

Name: ______________________________________

9. (5 pts) As we did in class, without writing out the math formulas, explain why $\binom{n}{k} = \binom{n}{n-k}$

10. (5 pts) Prove that for n ≥ 2, $\binom{n}{1} + \binom{n}{2} + \binom{n}{n-2} = n^2$ (use any convenient evaluation method for the choice symbol).

11. (10 pts) Write down the term in the expansion of $(3a^2 + 4b)^6$ in which a^6 appears.

12. (10 pts) Use the binomial theorem to compute $\sum_{k=0}^{5} 2^k \binom{5}{k}$

Name: ANSWER KEY

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1. (5 pts) Draw a *simple* graph with degree sequence (1,1,2,2,4).

2. (5 pts) Write as a summation, in sigma notation: $x/2 + x^2/6 + x^3/10 + x^4/14 + x^5/18$.

$$\sum_{j=1}^{5} \frac{x^j}{4j-2}$$

3. (5 pts) Evaluate the following summation. Be sure to *write out all the terms first*. You should end up with a fraction.

$$\sum_{i=0}^{5} (m^2 - 5m + 6)/(m+1)$$

$$\frac{0^2-5\cdot 0+6}{0+1} + \frac{1^2-5\cdot 1+6}{1+1} + \frac{2^2-5\cdot 2+6}{2+1} + \frac{3^2-5\cdot 3+6}{3+1} + \frac{4^2-5\cdot 4+6}{4+1} + \frac{5^2-5\cdot 5+6}{5+1}$$

$$\frac{6}{1} + \frac{2}{2} + 0 + 0 + \frac{2}{5} + \frac{6}{6}$$

$$= 6 + 1 + \frac{2}{5} + 1 = \frac{42}{5}$$

Name: ______________________________

4. (10 pts) Use induction to show that the following equality holds for all ~~n~~ M>=0. Remember to write out each part of the induction proof carefully.

$$\sum_{n=0}^{M} 3\,(n-2)\,(n+1) = (M+2)(M+1)(M-3)$$

Base Case $M=0$ $\quad 3(0-2)(0+1) = 3(-2)(1) = -6$

$(0+2)(0+1)(0-3) = 2(1)(-3) = -6$

✓

Inductive Hypothesis: IF $\displaystyle\sum_{n=0}^{M} 3(n-2)(n+1) = (M+2)(M+1)(M-3)$

Inductive Consequent: THEN $\displaystyle\sum_{n=0}^{M+1} 3(n-2)(n+1) = (M+1+2)(M+1+1)(M+1-3)$

Proof $\displaystyle\sum_{n=0}^{M+1} 3(n-2)(n+1) = \left[\sum_{n=0}^{M} 3(n-2)(n+1)\right] + 3(M+1-2)(M+1+1)$

Using the inductive hypothesis,

$= (M+2)(M+1)(M-3) + 3(M-1)(M+2)$

$= (M+2)\left[(M+1)(M-3) + 3(M-1)\right]$

$= (M+2)\left[M^2 - 2M - 3 + 3M - 3\right]$

$= (M+2)(M^2 + M - 6) =$

$= (M+2)(M+3)(M-2) = (M+3)(M+2)(M-2)$

as desired.

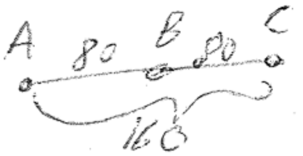
Conclusion $\displaystyle\sum_{n=0}^{M} 3(n-2)(n+1) = (M+2)(M+1)(M-3)$ for all $M \geq 0$

Name: ______________________________

5. For each one, state whether it is an equivalence relation or not. If it is not an equivalence relation, give all properties that fail (and tell why they fail):

   a. (5 pts) On the set of all persons born in the USA, let u~v if and only if u and v were born within one hundred kilometers of each other.

   Reflexive and Symmetric but not transitive



   $A \sim B$ But A and C are 160 km apart
   $B \sim C$ so $A \not\sim C$.

   b. (5 pts) On the set of all persons born in the USA, let u~v if and only if u and v were born in the same state.

   Reflexive, Symmetric, Transitive
   so Equivalence Relation

   c. (5 pts) On the set {a, b, c, d, e}, let u~v if and only if (u, v) is in the following set
   { (a,a), (b, a), (c,c), (a, c), (b, c), (c, b), (b, b), (c, a), (a, b), (d,d) }

   (e,e) missing so NOT reflexive
   (It is symmetric and transitive).

6. (10 pts) Use the definition of congruence modulo N to prove that if $j \equiv k$ (mod N) then $j^2 \equiv k^2$ (mod N).

$j \equiv k \bmod N$ means $j - k = N \cdot a$ some $a \in \mathbb{Z}$

Then $j = k + Na$ so $j^2 = (k + Na)^2 = k^2 + 2kNa + N^2a^2$

so $j^2 - k^2 = N[2ka + Na^2] = Nb$ for some $b \in \mathbb{Z}$

so by definition $j^2 \equiv k^2 \bmod N$.

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7. (10 pts) Use the ***standard*** (i.e., easy) Vigenère algorithm to decode the following message. You will find it helpful to have the following table.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
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	P	E	O	W	T	S	T	Q	P	E	O	W	T
	15	4	14	22	19	18	19	16	15	4	14	22	19
−	2	0	19	18	2	0	19	18	2	0	19	18	2
mod26	13	4	21	4	17	18	0	24	13	4	21	4	17
	N	E	V	E	R	S	A	Y	N	E	V	E	R

8. Consider the problem: In how many ways could you plan a pizza party, if you want exactly 5 different pizzas from a list of 7 pizza types, and you want exactly 3 types of sodas from a list of 5 soda types?

(a.) (4 pts) Express the solution in choice notation: $\binom{7}{5}\cdot\binom{5}{3}$

(b.) (4 pts) Show (at least) the relevant rows of Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

(c.) (2 pts) Indicate number of ways the ~~festival~~ party can be ~~formed~~ planned: $21\cdot 10 = 210$

Name: ________________________

9. (5 pts) As we did in class, without writing out the math formulas, explain why $\binom{n}{k} = \binom{n}{n-k}$

Choosing k out of n items is equivalent to getting rid of n-k out of n items.

10. (5 pts) Prove that for n ≥ 2, $\binom{n}{1} + \binom{n}{2} + \binom{n}{n-2} = n^2$ (use any convenient evaluation method for the choice symbol).

$\binom{n}{1} = \frac{n}{1}$ $\quad \binom{n}{2} = \binom{n}{n-2}$ as noted in previous problem

and $\binom{n}{2} = \frac{n(n-1)}{1 \cdot 2}$ $\quad$ So $\binom{n}{1} + \binom{n}{2} + \binom{n}{n-2} = n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$

$= n + n(n-1) = n^2$

11. (10 pts) Write down the term in the expansion of $(3a^2 + 4b)^6$ in which a^6 appears.

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Let $n=6$, $x = 3a^2$, $y = 4b$. Consider the term with $k=3$:

$\binom{6}{3}(3a^2)^3(4b)^{6-3} = 20 \cdot 3^3 \cdot a^6 \cdot 4^3 \cdot b^3 = (20 \cdot 3^3 \cdot 4^3) \cdot a^6 b^3$

12. (10 pts) Use the binomial theorem to compute $\sum_{k=0}^{5} 2^k \binom{5}{k}$

$\sum_{k=0}^{5} 2^k \binom{5}{k} = \sum_{k=0}^{5} \binom{5}{k} 2^k \cdot 1^{5-k} = (2+1)^5 = 3^5$

$= 243$