VILLANOVA UNIVERSITY
Department of Computing Sciences
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CSC 1300

Show your work carefully. Just writing an answer will not do. Show any assumptions; show the steps you took, and show how you came to your answer. Make sure your handwriting is clear and legible.
You may have the use of one index card with whatever information you chose to put on it.
No calculators, phones, or any other information sources are permitted.

Exam 2 of 3

From Exam 1

1. $f(n) = \left\lfloor \frac{n}{3} \right\rfloor + 5$. Domain is N, target space W
   i. *Find* $f(4) =$ 6

   ii. *Is* $f(n)$ $one-to-one$? (Explain)

   No. Multiple values of *n* map to a single value of *f(n)*.

   iii. Is f *onto?* (Explain)
   No. There are values of the target space to which no f(n) maps. (Example: 1, 2 ,..)

Chapter 4

2. Write out in full. (Write out all the terms. You do not need to simplify or calculate the value.)

$$\sum_{j=0}^{5} \frac{2j+1}{3}$$

$$= \frac{2(0)+1}{3} + \frac{2(1)+1}{3} + \frac{2(2)+1}{3} + \frac{2(3)+1}{3} + \frac{2(4)+1}{3} + \frac{2(5)+1}{3}$$

3. Show by induction that every tree is bipartite.

This was done in a Try This problem

4. Use induction to prove the following statement. Be sure to write out each step of the induction proof, properly labeled.

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Base: n = 2. $\sum_{i=1}^{2} i^3 = 1^3 + 2^3 = 9 \qquad \frac{2^2\,(2+1)^2}{4} = \frac{4*9}{4} = 9$

Induction Hypothesis: $\forall\, n \leq k$, *the statement is true*

Induction step. Consider n = k + 1

$\sum_{i=1}^{k+1} i^3$ Does this equal $\frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2\,(k+2)^2}{4}$

$\sum_{i=1}^{k} i^3 + (k+1)^3$

BY THE INDUCTION HYPOTHESIS

$\frac{k^2(k+1)^2}{4} + (k+1)^3$

$\frac{k^2(k+1)^2 + 4(k+1)^3}{4}$

$\frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4}$

$(k+1)^2(k^2 + 4(k+1))/4$

$\frac{(k+1)^2\,(k^2 + 4k + 4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$

MATCH

By the induction hypothesis, $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$

So we now have $\frac{k^2(k+1)^2}{4}$ + $(k+1)^3$

Chapter 5

5. Show that
   a. < is not an equivalence relation

      not reflexive ($a \nless a$)

   b. = is an equivalence relation

      reflexive $a=a$
      Symmetric if $a=b$, $b=a$
      Transitive if $a=b$ and $b=c$, $a=c$

   c. We can express an equivalence relation on a set by showing ordered pairs that are equivalent. If the given pairs meet all the required conditions, then we have an equivalence relation.
      Given the set {a, b, c, d} and the relation ~ such that x ~ y if and only if (x,y) is in the following set, is ~ an equivalence relation? Justify your answer by stating each requirement and illustrating that it is or is not met in this case.

      {(a,a), (b,a), (a,c), (b,c), (b,b), (c,c), (a,b), (d,d), (c,b)}

Reflexive (a,a) (b,b) (c,c) (d,d) ✓

Either shows not equivalence

Symmetric (b,a) (a,b)
(a,c) ⟷ no (c,a) fail

Transitive (c,b) and (b,a) no (c,a) fail

6. Congruence modulo n
   a. Write the formal definition of congruence modulo n: Complete this statement: $a \equiv b \bmod n$ is expressed as $a - b = kn$

   b. Using the formal definition of congruence modulo n, prove that if $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $a + c \equiv b + d \pmod n$

$a - b = k_1 n \qquad c - d = k_2 n$

$a - b + c - d = k_1 n + k_2 n$

$a + c - b - d = (k_1 + k_2) n$

$(a + c) - (b + d) = (k_1 + k_2) n$

Here is the promised chard of numerical values for the letters of the alphabet:

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

7. Use the ***standard*** Vigenère algorithm to decode the following message. Show all of your steps in the table provided.

   Keyword: **spring** Show coded keyword: ____________________

   Coded message: (You may choose not to use all the available lines. Do show your work clearly.)

e	p	i	k	u	s	s	s	e	m	f	y

Not this time

Chapter 6

8. Pascal's triangle
   a. Write out the first 7 rows of the Pascal triangle.
   b. Use the values in the triangle to show the number of ways to choose three flavors of ice cream from five choices. (First write the answer in choice notation, then find the answer in the triangle and circle it.)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1] For 10 a
1 8 28 56 70 56 28 8 1

$\binom{5}{3} = 10$

9. Write $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} \dots \pm \binom{n}{n}$ in summation notation. Then evaluate it, somehow. Explain what you are doing.

$\sum_{i=0}^{n} \binom{n}{i}(-1)^i = (x+y)^n$ where $x = -1$ $y = 1$

$(-1+1)^n = 0$

10. Quick binomial expansion questions
   a. Find the coefficient of the monomial containing d^4 in $(3c^2 - 5d)^8$

DO NOT DO A FULL EXPANSION OF THIS EXPRESSION. USE WHAT YOU HAVE LEARNED TO DO THIS EFFICIENTLY. You must show your work. Show the full coefficient, though you do not have to calculate the numeric value.

$x = 3c^2$
$y = -5d$
$(x+y)^8$

The term containing d^4 comes from y^4

$\binom{8}{4} x^4 y^4$

$70(x^4 y^4) = 70(3c^2)(-5d)^4$
$= 70 \cdot 3 \cdot (-5)^4 \cdot c^2 \cdot d^4$

   b. Use the binomial theorem to compute the value of the following expression. Be sure to show your work. This should be quick.

$$\sum_{i=0}^{m} 8^i (-7)^{m-i} \binom{m}{i}$$

$(x+y)^m$ $x = 8$ $y = -7$

$(8-7)^m = 1$